# Section 10

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# Administrivia

- Final PSet due yesterday (March 6)
- Final PSet coding due tomorrow (March 8)
- Final exam on March 11
    - Review Session during lecture tomorrow
    - See Anna's post on Edstem for more details
    - We're so close to the end of the quarter!! 🥳

# Review

# List of Topics

- Counting (Combinations, Stars and Bars, Pigeonhole Principle)
- Probability (Conditional Probability, LTP, Bayes Theorem, Chain Rule, Independence)
- Properties of RVs (PMFs, CDFs, Conversion from CDF to PDF)
- Expectation and Variance (LOTUS, Indicator RVs)
- Discrete Zoo RVs (Bernoulli, Binomial, Geometric, Poisson)
- Continuous RVs and Zoo RVs (PDF, CDF, Exponential, Normal,  CLT, Continuity Correction)
- Joint/Marginal Distributions

Note that this is not a comprehensive list of topics from this quarter, but rather a list of topics covered by problems on the section handout.

# Task 1a

a) **True or False:** The probability of getting 20 heads in 100 independent tosses of a coin that has probability 5/6 of coming up heads is $(5/6)^{20}(1/6)^{80}$.

# Task 1a - Solution

a) **True or False:** The probability of getting 20 heads in 100 independent tosses of a coin that has probability 5/6 of coming up heads is $(5/6)^{20}(1/6)^{80}$.

False. It is $\binom{100}{20}(5/6)^{20}(1/6)^{80}$. We need to account for the different orderings of the heads and the tails.

# Task 1b

b) **True or False:** Suppose we roll a six-sided fair die twice independently. Then the event that the first roll is 3 and the sum of the two rolls is 6 are independent.

# Task 1b - Solution

**b) True or False:** Suppose we roll a six-sided fair die twice independently. Then the event that the first roll is 3 and the sum of the two rolls is 6 are independent.

False. Let $X_1$ and $X_2$ be random variables that represent the values of the first and second rolls, respectively. $P(X_1 = 3) = \frac{1}{6}$. However, $P(X_1 = 3 \mid X_1 + X_2 = 6) = \frac{1}{5}$

# Task 1c

c) **True or False:** If $X$ and $Y$ are nonnegative, discrete, and independent random variables, then so are $X^2$ and $Y^2$.

# Task 1c - Solution

c) **True or False:** If $X$ and $Y$ are nonnegative, discrete, and independent random variables, then so are $X^2$ and $Y^2$.

True. $X^2$ and $Y^2$ are independent if $\mathbb{P}(X^2 = x, Y^2 = y) = \mathbb{P}(X^2 = x)\mathbb{P}(Y^2 = y)$.

$\mathbb{P}(X^2 = x, Y^2 = y) = \mathbb{P}(X = \sqrt{x}, Y = \sqrt{y})$ and since $X$ and $Y$ are independent:

$\mathbb{P}(X = \sqrt{x}, Y = \sqrt{y}) = \mathbb{P}(X = \sqrt{x})\mathbb{P}(Y = \sqrt{y}) = \mathbb{P}(X^2 = x)\mathbb{P}(Y^2 = y)$ Thus, $X^2$ and $Y^2$ are independent.

# Task 1d

d) **True or False:** The central limit theorem requires the random variables to be independent.

# Task 1d - Solution

**d) True or False:** The central limit theorem requires the random variables to be independent.

True. The central limit theorem requires the random variables to be i.i.d.

# Task 1e

e) **True or False:** Let $A$, $B$ and $C$ be any three events defined with respect to a probability space. Then $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A \cap B|C)\mathbb{P}(B|C)\mathbb{P}(C)$.

# Task 1e - Solution

e) **True or False:** Let $A$, $B$ and $C$ be any three events defined with respect to a probability space. Then $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A \cap B|C)\mathbb{P}(B|C)\mathbb{P}(C)$.

False. Suppose $A$, $B$, and $C$ are all mutually independent, then

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C) \neq \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(B)\,\mathbb{P}(C) = \mathbb{P}(A \cap B \mid C)\mathbb{P}(B \mid C)\mathbb{P}(C)$$

In general, one correct way to apply the chain rule (twice) would be

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A \mid B \cap C)\mathbb{P}(B \cap C) = \mathbb{P}(A \mid B \cap C)\mathbb{P}(B \mid C)\mathbb{P}(C)$$

# Task 1f

**f)** **True or False:** If you flip a fair coin 1000 times, then the probability that there are 800 heads in total is the same as the probability that there are 80 heads in the first 100 flips.

# Task 1f - Solution

f) **True or False:** If you flip a fair coin 1000 times, then the probability that there are 800 heads in total is the same as the probability that there are 80 heads in the first 100 flips.

False. Let $X$ be the number of heads in 1000 flips of a fair coin, and Let $Y$ be the number of heads in 100 flips of a fair coin.

$$\mathbb{P}(X = 800) = \binom{1000}{800} 0.5^{1000} = 6.17 \cdot 10^{-86} \neq 4.22 \cdot 10^{-10} = \binom{100}{80} 0.5^{100}$$

# Task 1g

**g) True or False:** If $N$ is a nonnegative integer valued random variable, then

$$\mathbb{E}\left[\binom{N}{2}\right] = \binom{\mathbb{E}[N]}{2}.$$

# Task 1g - Solution

g) **True or False:** If $N$ is a nonnegative integer valued random variable, then

$$\mathbb{E}\left[\binom{N}{2}\right] = \binom{\mathbb{E}[N]}{2}.$$

False. The left-hand side is

$$\mathbb{E}[\binom{N}{2}] = \mathbb{E}[\frac{N!}{(N-2)!\,2!}] = \frac{1}{2}\,\mathbb{E}[N^2 - N] = \frac{1}{2}\,(\mathbb{E}[N^2] - \mathbb{E}[N])$$

while the right-hand side is

$$\binom{\mathbb{E}[N]}{2} = \frac{\mathbb{E}[N]!}{(\mathbb{E}[N]-2)!\,2!} = \frac{1}{2}\,(\mathbb{E}[N]^2 - \mathbb{E}[N])$$

and in general these equations are not equal because $\mathbb{E}[N^2] \neq \mathbb{E}[N]^2$

# Task 1a

a) Consider a set $S$ containing $k$ distinct integers. What is the smallest $k$ for which $S$ is guaranteed to have 3 numbers that are the same mod 5?

# Task 1a - Solution

**a)** Consider a set $S$ containing $k$ distinct integers. What is the smallest $k$ for which $S$ is guaranteed to have 3 numbers that are the same mod 5?

$k = 11$. This is because modding any number by 5 yields 5 possible integers (i.e. slots). When distributing 11 numbers between these five slots, one slot must correspond to at least 3 integers mod 5.

# Task 1b

**b)** Let $X$ be a discrete random variable that can only be between -10 and 10. That is, $P(X = x) \geqslant 0$ for $-10 \leqslant x \leqslant 10$, and $P(X = x) = 0$ otherwise. What is the smallest possible value the variance of $X$ can take?

# Task 2b - Solution

**b)** Let $X$ be a discrete random variable that can only be between -10 and 10. That is, $P(X = x) \geqslant 0$ for $-10 \leqslant x \leqslant 10$, and $P(X = x) = 0$ otherwise. What is the smallest possible value the variance of $X$ can take?

0. This is because $Var(X) \geqslant 0$ and we can define the probability mass function in a way makes $Var(X) = 0$, as long as the support of $X$ lies between -10 and 10. For example, we can define a PMF $p_X(x) = 1$ if $x = 7$ and 0 otherwise. Then we have

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 7^2 - 7^2 = 0$$

# Task 1c

c) How many ways are there to rearrange the letters in the word KNICKKNACK?

# Task 1c - Solution

c) How many ways are there to rearrange the letters in the word KNICKKNACK?

$\frac{10!}{4!\,2!\,2!}$. Permute all 10 letters as if distinct, then divide by 4! to account for over counting the Ks; divide by 2! to account for over counting the Cs; and divide by 2! again to account for over counting the Ns

# Task 1d

**d)** I toss n balls into n bins uniformly at random. What is the expected number of bins with exactly $k$ balls in them?

# Task 1d - Solution

**d)** I toss n balls into n bins uniformly at random. What is the expected number of bins with exactly $k$ balls in them?

Let $X$ be the number of bins with $k$ balls in them. Let $X_i$ be 1 if the $i$th bin has $k$ balls in it, and otherwise 0. Note that $X = \sum_{i=1}^{n} X_i$. Since balls are distributed uniformly at random, the probability that a particular ball lands in a particular bin is $1/n$. Thus, the probability that $k$ balls land in the $i$th bin is $\binom{n}{k}\left(\frac{1}{n}\right)^k\left(\frac{n-1}{n}\right)^{n-k}$. By linearity of expectation we have

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \mathbb{P}(X_i = 1) = \sum_{i=1}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k\left(\frac{n-1}{n}\right)^{n-k} = n\binom{n}{k}\left(\frac{1}{n}\right)^k\left(\frac{n-1}{n}\right)^{n-k}$$

# Task 1e

e) Consider a six-sided die where $Pr(1) = Pr(2) = Pr(3) = Pr(4) = 1/8$ and $Pr(5) = Pr(6) = 1/4$. Let $X$ be the random variable which is the square root of the value showing. (For example, if the die shows a 1, $X$ is 1, if the die shows a 2, $X$ is $\sqrt{2}$, if the die shows a 3, $X = \sqrt{3}$ and so on.) What is the expected value of $X$? (Leave your answer in the form of a numerical sum; do not bother simplifying it.)

# Task 1e - Solution

**e)** Consider a six-sided die where $Pr(1) = Pr(2) = Pr(3) = Pr(4) = 1/8$ and $Pr(5) = Pr(6) = 1/4$. Let $X$ be the random variable which is the square root of the value showing. (For example, if the die shows a 1, $X$ is 1, if the die shows a 2, $X$ is $\sqrt{2}$, if the die shows a 3, $X = \sqrt{3}$ and so on.) What is the expected value of $X$? (Leave your answer in the form of a numerical sum; do not bother simplifying it.)

By the definition of expectation

$$\mathbb{E}[X] = \sum_{x=1}^{6} \sqrt{x}\, \mathbb{P}(x)$$

# Task 1f

f) A bus route has interarrival times that are exponentially distributed with parameter $\lambda = \frac{0.05}{\text{min}}$. What is the probability of waiting an hour or more for a bus?

# Task 1f - Solution

f) A bus route has interarrival times that are exponentially distributed with parameter $\lambda = \frac{0.05}{\text{min}}$. What is the probability of waiting an hour or more for a bus?

Let $X$ be an RV representing wait time, distributed according to $Exp(0.05)$

$$\mathbb{P}(X > 60) = 1 - F_X(60) = 1 - (1 - e^{-0.05 \cdot 60}) = 0.0498$$

# Task 1g

g) How many different ways are there to select 3 dozen indistinguishable colored roses if red, yellow, pink, white, purple and orange roses are available?

# Task 1g - Solution

g) How many different ways are there to select 3 dozen indistinguishable colored roses if red, yellow, pink, white, purple and orange roses are available?

This is just a stars and bars problem. In this case there are 36 stars and 5 bars. So there are $\binom{41}{5}$ ways to select 3 dozen roses.

# Task 1h

**h)** Two identical 52-card decks are mixed together. How many distinct permutations of the 104 cards are there?

# Task 1h - Solution

**h)** Two identical 52-card decks are mixed together. How many distinct permutations of the 104 cards are there?

Perform the permutation as if it were 104 distinct items, and divide out the duplicates (each pair has 2! excess orderings, and there are 52 pairs), to get:

$$\frac{104!}{(2!)^{52}}$$

# Task 2

Consider a boolean formula on $n$ variables in 3-CNF, that is, conjunctive normal form with 3 literals per clause. This means that it is an "and" of "ors", where each "or" has 3 literals. Each parenthesized expression (i.e., each "or" of three literals) is called a clause. Here is an example of a boolean formula in 3-CNF, with $n = 6$ variables and $m = 4$ clauses.

$$(x_1 \vee x_3 \vee x_5) \wedge (\neg x_1 \vee \neg x_2 \vee x_6) \wedge (x_5 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee x_4 \vee x_5).$$

**a)** What is the probability that $(\neg x_1 \vee \neg x_2 \vee x_3)$ evaluates to true if variable $x_i$ is set to true with probability $p_i$, independently for all $i$?

**b)** Consider a boolean formula in 3-CNF with $n$ variables and $m$ clauses, where the three literals in each clause refer to distinct variables. What is the expected number of satisfied clauses if each variable is set to true independently with probability 1/2? A clause is satisfied if it evaluates to true. (In the displayed example above, if $x_1, \dots, x_5$ are set to true and $x_6$ is set to false, then all clauses but the second are satisfied.)

# Task 2a - Solution

**a)** What is the probability that $(\neg x_1 \vee \neg x_2 \vee x_3)$ evaluates to true if variable $x_i$ is set to true with probability $p_i$, independently for all $i$?

$(\neg x_1 \vee \neg x_2 \vee x_3)$ is true when at least one of the following holds: $x_1 = \texttt{false}$, $x_2 = \texttt{false}$, $x_3 = \texttt{true}$. So we can write

$$\begin{aligned}
\mathbb{P}((\neg x_1 \vee \neg x_2 \vee x_3) = \texttt{true}) &= \mathbb{P}(x_1 = \texttt{false} \cup x_2 = \texttt{false} \cup x_3 = \texttt{true}) \\
&= 1 - \mathbb{P}(x_1 = \texttt{true} \cap x_2 = \texttt{true} \cap x_3 = \texttt{false}) && \text{[Complementary probability]} \\
&= 1 - \mathbb{P}(x_1 = \texttt{true})\mathbb{P}(x_2 = \texttt{true})\mathbb{P}(x_3 = \texttt{false}) && \text{[Independence]} \\
&= 1 - p_1 \cdot p_2 \cdot (1 - p_3)
\end{aligned}$$

# Task 2b - Solution

**b)** Consider a boolean formula in 3-CNF with $n$ variables and $m$ clauses, where the three literals in each clause refer to distinct variables. What is the expected number of satisfied clauses if each variable is set to true independently with probability 1/2? A clause is satisfied if it evaluates to true. (In the displayed example above, if $x_1, \ldots, x_5$ are set to true and $x_6$ is set to false, then all clauses but the second are satisfied.)

Let $X$ be a random variable that represents the total number of satisfied clauses. Let $X_i$ be a random variable that is 1 if the $i$th clause is satisfied, and otherwise 0. Note that $X = \sum_{i=1}^{m} X_i$. The $\mathbb{P}(X_i = 1) = 1 - 0.5^3$. This is because the $i$th clause is true when at least one of its disjuncts evaluates to true. As discussed in the previous part, this is equivalent to not all disjuncts evaluating to false. The probability that an individual disjuncts evaluates to false is 0.5, and because each conjuncts truth value is independent of the others, the probability that they are all false is $0.5^3$. Using the complementary probability rule, we get $\mathbb{P}(X_i = 1) = 1 - 0.5^3$. By linearity of expectation

$$\mathbb{E}[X] = \sum_{i=1}^{m} \mathbb{E}[X_i] = \sum_{i=1}^{m} \mathbb{P}(X_i = 1) = \sum_{i=1}^{m} (1 - 0.5^3) = m\left(1 - 0.5^3\right)$$

# Task 3

We flip a biased coin with probability $p$ of getting heads until we either get heads or we flip the coin three times. Thus, the possible outcomes of this random experiment are $< H >, < T, H >, < T, T, H >$ and $< T, T, T >$.

**a)** What is the probability mass function of $X$, where $X$ is the number of heads. (Notice that $X$ is 1 for the first three outcomes, and 0 in the last outcome.)

**b)** What is the probability that the coin is flipped more than once?

**c)** Are the events "there is a second flip and it is heads" and "there is a third flip and it is heads" independent? Justify your answer.

**d)** Given that we flipped more than once and ended up with heads, what is the probability that we got heads on the second flip? (No need to simplify your answer.)

# Task 3a - Solution

a) What is the probability mass function of $X$, where $X$ is the number of heads. (Notice that $X$ is 1 for the first three outcomes, and 0 in the last outcome.)

Let $E$ be an event that represents the outcome of our experiment. Note that $E$ can take on four possible outcomes, however, they do not occur with equal probability.

$$\begin{aligned}\mathbb{P}(X=0) &= \mathbb{P}(E = < T, T, T >) \\ &= (1-p)^3 && \text{[Independent flips]}\end{aligned}$$

And

$$\begin{aligned}\mathbb{P}(X=1) &= 1 - \mathbb{P}(X=0) && \text{[Complementing]} \\ &= 1-(1-p)^3\end{aligned}$$

# Task 3a - Solution

a) What is the probability mass function of $X$, where $X$ is the number of heads. (Notice that $X$ is 1 for the first three outcomes, and 0 in the last outcome.)

Alternatively, we can calculate $\mathbb{P}(X = 1)$ as

$$\begin{aligned}
\mathbb{P}(X = 1) &= \mathbb{P}(E =< H > \cup E =< T, H > \cup E =< T, T, H >) \\
&= \mathbb{P}(E =< H >) + \mathbb{P}(E =< T, H >) + \mathbb{P}(E =< T, T, H >) && \text{[Disjoint events]} \\
&= p + (1-p)p + (1-p)^2 p && \text{[Independent flips]}
\end{aligned}$$

Thus,

$$p_X(x) = \begin{cases} (1-p)^3, & x = 0 \\ p + (1-p)p + (1-p)^2 p, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

# Task 3b - Solution

**b)** What is the probability that the coin is flipped more than once?

The coin is flipped more than once if $E$ is any of the last three outcomes. This is equivalent to $E$ not being the first outcome. This occurs with probability $1 - \mathbb{P}(E =< H >) = 1 - p$.

# Task 3c - Solution

c) Are the events "there is a second flip and it is heads" and "there is a third flip and it is heads" independent? Justify your answer.

The event "there is a second flip and it is heads" is independent from the event "there is a third flip and it is heads" if and only if the following equation holds:

$$\mathbb{P}(E =< T, H >| E =< T, T, H >) = \mathbb{P}(E =< T, H >)$$

The LHS is 0 because it is impossible to flip $T, H$ if you've already flipped $T, T, H$, whereas the RHS is $(1-p)p$. Therefore, the events are not independent.

# Task 3d - Solution

**d)** Given that we flipped more than once and ended up with heads, what is the probability that we got heads on the second flip? (No need to simplify your answer.)

Given that we flipped more than once and ended up with heads means that

$$E =< T, H > \cup E =< T, T, H >$$

Now, we are trying to find the following probability: $\mathbb{P}(E =< T, H >| (E =< T, H > \cup E =< T, T, H >))$.
By the definition of conditional probability this is equal to

$$\frac{\mathbb{P}(E =< T, H > \cap (E =< T, H > \cup E =< T, T, H >))}{\mathbb{P}(E =< T, H > \cup E =< T, T, H >)} = \frac{\mathbb{P}(E =< T, H >)}{\mathbb{P}(E =< T, H > \cup E =< T, T, H >)}$$
$$= \frac{(1-p)p}{(1-p)p + (1-p)^2 p}$$

The first equality holds because $E =< T, H >$ and $E =< T, T, H >$ are disjoint events, and the second equality holds from the probability values of the event $E$ that we found in part (a).

# Task 4

There is a population of $n$ people. The number of Bitcoin users among these $n$ people is $i$ with probability $p_i$, where, of course, $\sum_{0 \leqslant i \leqslant n} p_i = 1$. We take a random sample of $k$ people from the population (without replacement). Use Bayes Theorem to derive an expression for the probability that there are $i$ Bitcoin users in the population conditioned on the fact that there are $j$ Bitcoin users in the sample. Let $B_i$ be the event that there are $i$ Bitcoin users in the population and let $S_j$ be the event that there are $j$ Bitcoin users in the sample. Your answer should be written in terms of the $p_\ell$'s, $i, j, n$ and $k$. Your answer can contain summation notation.

# Task 4 - Sol.

There is a population of $n$ people. The number of Bitcoin users among these $n$ people is $i$ with probability $p_i$, where, of course, $\sum_{0 \leqslant i \leqslant n} p_i = 1$. We take a random sample of $k$ people from the population (without replacement). Use Bayes Theorem to derive an expression for the probability that there are $i$ Bitcoin users in the population conditioned on the fact that there are $j$ Bitcoin users in the sample. Let $B_i$ be the event that there are $i$ Bitcoin users in the population and let $S_j$ be the event that there are $j$ Bitcoin users in the sample. Your answer should be written in terms of the $p_\ell$'s, $i, j, n$ and $k$. Your answer can contain summation notation.

$$
\begin{aligned}
Pr(B_i|S_j) &= \frac{Pr(S_j|B_i)Pr(B_i)}{Pr(S_j)} \qquad \text{by Bayes Theorem} \\
&= \frac{\frac{\binom{i}{j}\binom{n-i}{k-j}}{\binom{n}{k}} \cdot p_i}{\sum_{\ell=0}^{n} Pr(S_j|B_\ell)Pr(B_\ell)} = \frac{\frac{\binom{i}{j}\binom{n-i}{k-j}}{\binom{n}{k}} \cdot p_i}{\sum_{\ell=0}^{n} \frac{\binom{\ell}{j}\binom{n-\ell}{k-j}}{\binom{n}{k}} \cdot p_\ell} = \frac{\binom{i}{j}\binom{n-i}{k-j} \cdot p_i}{\sum_{\ell=0}^{n} \binom{\ell}{j}\binom{n-\ell}{k-j} \cdot p_\ell}.
\end{aligned}
$$

Above, we used the fact that $Pr(B_\ell) = p_\ell$ and the fact that $Pr(S_j|B_\ell)$ is the probability of choosing a subset of size $k$, where $j$ of the selected people are from the subset of $\ell$ Bitcoin users and $k-j$ are from the remaining $n-\ell$ non-Bitcoin users. That is, $Pr(S_j|B_\ell)$ is the probability of drawing the number $j$ from a HyperGeometric($n, i, k$) random variable.

# Task 5

You are considering three investments. Investment A yields a return which is $X$ dollars where $X$ is Poisson with parameter 2. Investment B yields a return of $Y$ dollars where $Y$ is Geometric with parameter 1/2. Investment C yields a return of $Z$ dollars which is Binomial with parameters $n = 20$ and $p = 0.1$. The returns of the three investments are independent.

**a)** Suppose you invest simultaneously in all three of these possible investments. What is the expected value and the variance of your total return?

**b)** Suppose instead that you choose uniformly at random from among the 3 investments (i.e., you choose each one with probability 1/3). Use the law of total probability to write an expression for the probability that the return is 10 dollars. Your final expression should contain numbers only. No need to simplify your answer.

# Task 5a - Solution

a) Suppose you invest simultaneously in all three of these possible investments. What is the expected value and the variance of your total return?

a) Let $R$ be a random variable representing the total returns you get. If we invest in all of them simultaneously, then $R = X + Y + Z$. Then, $\mathbb{E}[R] = \mathbb{E}[X + Y + Z] = \mathbb{E}[X] + \mathbb{E}[Y] + \mathbb{E}[Z]$ by linearity of expectation.

Since $X$ is Poisson with parameter 2, $\mathbb{E}[X] = 2$. $Y$ is Geometric with parameter $\frac{1}{2}$, so $\mathbb{E}[Y] = \frac{1}{1/2} = 2$. $Z$ is Binomial with parameters $n = 20$ and $p = 0.1$, so $\mathbb{E}[Z] = 20 \cdot 0.1 = 2$. Thus $\mathbb{E}[R] = 2 + 2 + 2 = 6$

$Var(R) = Var(X+Y+Z) = Var(X)+Var(Y)+Var(Z)$ because the returns from all three investments are independent. Because we know the distributions, we can read off their variances, with $Var(X) = \lambda = 2, Var(Y) = \frac{1-p}{p^2} = \frac{1/2}{1/4} = 2, Var(Z) = np(1-p) = 20 \cdot 0.1(0.9) = 1.8$.

Thus, $Var(R) = 2 + 2 + 1.8 = 5.8$

# Task 5b - Solution

**b)** Suppose instead that you choose uniformly at random from among the 3 investments (i.e., you choose each one with probability 1/3). Use the law of total probability to write an expression for the probability that the return is 10 dollars. Your final expression should contain numbers only. No need to simplify your answer.

**b)** Define events $A, B,$ and $C$ as randomly choosing Investments A, B, and C respectively. We want to find $\mathbb{P}(R = 10)$. We can break this up with the Law of Total Probability as

$$\mathbb{P}(R = 10) = \mathbb{P}(R = 10|A)(\frac{1}{3}) + \mathbb{P}(R = 10|B)(\frac{1}{3}) + \mathbb{P}(R = 10|C)(\frac{1}{3})$$

In each case, $R = X, Y,$ or $Z$ respectively, so we can plug in the PMFs of each function (and distribute out the $\frac{1}{3}$):

$$\mathbb{P}(R = 10) = \frac{1}{3}(e^{-2}\frac{2^{10}}{10!} + (0.5)^9 \cdot 0.5 + \binom{20}{10}0.1^{10}(0.9)^{10}) = 3.4040 \cdot 10^{-4}$$

# Task 6

The density function of $X$ is given by

$$f(x) = \begin{cases} a + bx^2 & \text{when } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathbb{E}[X] = \frac{3}{5}$, find $a$ and $b$.

# Task 6 - Solution

The density function of $X$ is given by

$$f(x) = \begin{cases} a + bx^2 & \text{when } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathbb{E}[X] = \frac{3}{5}$, find $a$ and $b$.

To find the value of two variables, we need two equations to solve as a system. We know that $\mathbb{E}[X] = \frac{3}{5}$, so we know, by the definition of expected value, that

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx = \frac{3}{5}$$

# Task 6 - Solution

The density function of $X$ is given by

$$f(x) = \begin{cases} a + bx^2 & \text{when } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathbb{E}[X] = \frac{3}{5}$, find $a$ and $b$.

Since $f(x)$ is defined to be 0 outside of the given range, we can integrate within only that range, plugging in $f(x)$:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{0} xf(x)dx + \int_{0}^{1} xf(x)dx + \int_{1}^{\infty} xf(x)dx = \int_{0}^{1} x(a + bx^2)dx$$

$$= \int_{0}^{1} (ax + bx^3)dx = \frac{ax^2}{2} + \frac{bx^4}{4}\Big|_0^1 = \frac{a}{2} + \frac{b}{4} = \frac{3}{5}$$

We also know that a valid density function integrates to 1 over all possible values. Thus, we can perform the same process to get a second equation:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} xf(x)dx + \int_{0}^{1} xf(x)dx + \int_{1}^{\infty} xf(x)dx = \int_{0}^{1} (a+bx^2)dx = ax + \frac{bx^3}{3}\Big|_0^1 = a + \frac{b}{3} = 1$$

Solving this system of equations we get that $a = \frac{3}{5}, b = \frac{6}{5}$

# Task 7

A point is chosen at random on a line segment of length $L$. Interpret this statement and find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

# Task 7 - Solution

A point is chosen at random on a line segment of length $L$. Interpret this statement and find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

Define RV $X$ to be the distance of your random point from the leftmost side of the stick. Since we're choosing a point at random, this RV has an equal likelihood of any distance from 0 to $L$, making it a continuous uniform RV with parameters $a = 0, b = L$. For the ratio to be less than $\frac{1}{4}$, the shorter segment has to be less than $\frac{L}{5}$ in length.

This can happen when $X < \frac{L}{5}$ or $X > \frac{4L}{5}$. Thus, using the CDF of a continuous uniform distribution, the probability that the ratio is less than $\frac{1}{4}$ is

$$\mathbb{P}(X \leqslant \frac{L}{5}) + \mathbb{P}(X > \frac{4L}{5}) = F_X(\frac{L}{5}) + (1 - F_X(\frac{4L}{5})) = \frac{\frac{L}{5} - 0}{L - 0} + (1 - \frac{\frac{4L}{5} - 0}{L - 0}) = \frac{1}{5} + (1 - \frac{4}{5}) = \frac{2}{5}$$

# Task 8

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables each with CDF $F_X(x)$ and pdf $f_X(x)$. Let $Y = \min(X_1, \ldots, X_n)$ and let $Z = \max(X_1, \ldots, X_n)$. Show how to write the CDF and pdf of $Y$ and $Z$ in terms of the functions $F_X(\cdot)$ and $f_X(\cdot)$.

# Task 8 - Solution

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables each with CDF $F_X(x)$ and pdf $f_X(x)$. Let $Y = \min(X_1, \ldots, X_n)$ and let $Z = \max(X_1, \ldots, X_n)$. Show how to write the CDF and pdf of $Y$ and $Z$ in terms of the functions $F_X(\cdot)$ and $f_X(\cdot)$.

First we compute the CDFs of $Z$ and $Y$ as follows:

$$
\begin{aligned}
F_Z(z) &= P(Z < z) \\
&= P(X_1 < z, \ldots, X_n < z) && \text{[Definition of max]} \\
&= P(X_1 < z) \cdot \ldots \cdot P(X_n < z) && \text{[Independence]} \\
&= (F_X(z))^n
\end{aligned}
$$

$$
\begin{aligned}
F_Y(y) &= P(Y < y) \\
&= 1 - P(Y > y) \\
&= 1 - P(X_1 > y, \ldots, X_n > y) && \text{[Definition of min]} \\
&= 1 - P(X_1 > y) \cdot \ldots \cdot P(X_n > y) && \text{[Independence]} \\
&= 1 - (1 - F_X(y))^n
\end{aligned}
$$

# Task 8 - Solution

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables each with CDF $F_X(x)$ and pdf $f_X(x)$. Let $Y = \min(X_1, \ldots, X_n)$ and let $Z = \max(X_1, \ldots, X_n)$. Show how to write the CDF and pdf of $Y$ and $Z$ in terms of the functions $F_X(\cdot)$ and $f_X(\cdot)$.

Using the fact that $f_X(x) = \frac{d}{dx}F_X(x)$ and the CDFs that we found we can compute the pdfs of $Z$ and $Y$ as follows:

$$\begin{aligned} f_Z(z) &= \frac{d}{dz}F_Z(z) \\ &= \frac{d}{dz}(F_X(z))^n \\ &= n \cdot F_X(z)^{n-1} \cdot \left(\frac{d}{dz}F_X(z)\right) \\ &= n \cdot F_X(z)^{n-1} \cdot f_X(z) \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) \\ &= \frac{d}{dy}(1-(1-F_X(y))^n) \\ &= -n \cdot (1-F_X(y))^{n-1} \cdot \frac{d}{dy}(1-F_X(y)) \\ &= n \cdot (1-F_X(y))^{n-1} \cdot f_X(y) \end{aligned}$$

# Task 9

Let $X$ be the sum of 100 real numbers, and let $Y$ be the same sum, but with each number rounded to the nearest integer before summing. If the roundoff errors are independent and uniformly distributed between -0.5 and 0.5, what is the approximate probability that $|X - Y| > 3$?

# Task 9 - Solution

Let $X$ be the sum of 100 real numbers, and let $Y$ be the same sum, but with each number rounded to the nearest integer before summing. If the roundoff errors are independent and uniformly distributed between -0.5 and 0.5, what is the approximate probability that $|X - Y| > 3$?

Let $X = \sum_{i=1}^{100} X_i$, and $Y = \sum_{i=1}^{100} r(X_i)$, where $r(X_i)$ is $X_i$ rounded to the nearest integer. Then, we have

$$X - Y = \sum_{i=1}^{100} X_i - r(X_i)$$

Note that each $X_i - r(X_i)$ is simply the round off error, which is distributed as $Unif(-0.5, 0.5)$. Since $X - Y$ is the sum of 100 i.i.d. random variables with mean $\mu = 0$ and variance $\sigma^2 = \frac{1}{12}$,

# Task 9 - Solution

Let $X$ be the sum of 100 real numbers, and let $Y$ be the same sum, but with each number rounded to the nearest integer before summing. If the roundoff errors are independent and uniformly distributed between -0.5 and 0.5, what is the approximate probability that $|X - Y| > 3$?

$X - Y \approx W \sim \mathbb{N}(0, \frac{100}{12})$ by the Central Limit Theorem. For notational convenience let $Z \sim \mathbb{N}(0, 1)$

$$\begin{aligned}
\mathbb{P}(|X - Y| > 3) &\approx \mathbb{P}(|W| > 3) && \text{[CLT]} \\
&= \mathbb{P}(W > 3) + \mathbb{P}(W < -3) && \text{[No overlap between } W > 3 \text{ and } W < -3\text{]} \\
&= 2\,\mathbb{P}(W > 3) && \text{[Symmetry of normal]} \\
&= 2\,\mathbb{P}\left(\frac{W}{\sqrt{100/12}} > \frac{3}{\sqrt{100/12}}\right) \\
&\approx 2\,\mathbb{P}(Z > 1.039) && \text{[Standardize } W\text{]} \\
&= 2\,(1 - \Phi(1.039)) \approx 0.29834
\end{aligned}$$

# Task 10

A prolific twitter user tweets approximately 350 tweets per week. Let's assume for simplicity that the tweets are independent, and each consists of a uniformly random number of characters between 10 and 140. (Note that this is a discrete uniform distribution.) Thus, the central limit theorem (CLT) implies that the number of characters tweeted by this user is approximately normal with an appropriate mean and variance. Assuming this normal approximation is correct, estimate the probability that this user tweets between 26,000 and 27,000 characters in a particular week. (This is a case where continuity correction will make virtually no difference in the answer, but you should still use it to get into the practice!).

# Task 10 - Solution

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Let $X$ be the total number of characters tweeted by a twitter user in a week. Let $X_i \sim Unif(10, 140)$ be the number of characters in the $i$th tweet (since the start of the week). Since $X$ is the sum of 350 i.i.d. rvs with mean $\mu = 75$ and variance $\sigma^2 = 1430$, $X \approx N = \mathbb{N}(350 \cdot 75, 350 \cdot 1430)$. Thus,

$$\mathbb{P}(26,000 \leqslant X \leqslant 27,000) \approx \mathbb{P}(25,999.5 \leqslant N \leqslant 27,000.5)$$

# Task 10 - Solution

Standardizing this gives the following formula

$$\begin{aligned}\mathbb{P}(25,999.5 \leqslant N \leqslant 27,000.5) &\approx \mathbb{P}\left(-0.3541 \leqslant \frac{N - 350 \cdot 75}{\sqrt{350 \cdot 1430}} \leqslant 1.0608\right)\\ &= \mathbb{P}\left(-0.3541 \leqslant Z \leqslant 1.0608\right)\\ &= \Phi(1.0608) - \Phi(-0.3541)\\ &\approx 0.4923\end{aligned}$$

So the probability that this user tweets between 26,000 and 27,000 characters in a particular week is approximately 0.4923.

# Task 11

A delivery guy in some company is out delivering $n$ packages to $n$ customers, where $n \in N, n > 1$. Not only does he hand each customer a package uniformly at random from the remaining packages, he opens the package before delivering it with probability $\frac{1}{2}$. Let $X$ be the number of customers who receive their own packages unopened.

a) Compute the expectation $\mathbb{E}[X]$.

b) Compute the variance $\mathrm{Var}(X)$.

# Task 11 - Solution

A delivery guy in some company is out delivering $n$ packages to $n$ customers, where $n \in N, n > 1$. Not only does he hand each customer a package uniformly at random from the remaining packages, he opens the package before delivering it with probability $\frac{1}{2}$. Let $X$ be the number of customers who receive their own packages unopened.

**a)** Compute the expectation $\mathbb{E}[X]$.

Let $X_i$ be an indicator random variable where $X_i = 1$ if the $i_{\text{th}}$ customer gets their packaged unopened, and $X_i = 0$ otherwise. So, we have that $X = \sum_{i=1}^{n} X_i$. By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbb{E}[X_i]$$

Since $X_i$ is a Bernoulli random variable, we have

$$\mathbb{E}[X_i] = \mathbb{P}(X_i = 1) = \frac{1}{2n}$$

since the $i^{\text{th}}$ customer will get their own package with probability $\frac{1}{n}$ and it will be unopened with probability $\frac{1}{2}$, and the delivery guy opens the packages independently. Hence, $\mathbb{E}[X] = n \cdot \frac{1}{2n} = \frac{1}{2}$.

## Task 11 - Solution

A delivery guy in some company is out delivering $n$ packages to $n$ customers, where $n \in N, n > 1$. Not only does he hand each customer a package uniformly at random from the remaining packages, he opens the package before delivering it with probability $\frac{1}{2}$. Let $X$ be the number of customers who receive their own packages unopened.

To calculate $\mathrm{Var}(X)$, we need to find $\mathbb{E}[X^2]$. By Linearity of Expectation,

$$\mathbb{E}[X^2] = \mathbb{E}[(X_1 + X_2 + \ldots + X_n)^2] = \mathbb{E}[\sum_{i,j} X_i X_j] = \sum_{i,j} \mathbb{E}[X_i X_j]$$

Then, we consider two cases, either $i = j$ or $i \neq j$. Hence, $\sum_{i,j} \mathbb{E}[X_i X_j] = \sum_i \mathbb{E}[X_i^2]$ $\sum_{i \neq j} \mathbb{E}[X_i X_j]$. So, by LOTUS, we have for all $i$,

$$\mathbb{E}[X_i^2] = 1^2 \cdot \mathbb{P}(X_i = 1) + 0^2 \cdot \mathbb{P}(X_i = 0) = \mathbb{E}[X_i] = \frac{1}{2n}$$

# Task 11 - Solution

A delivery guy in some company is out delivering $n$ packages to $n$ customers, where $n \in N, n > 1$. Not only does he hand each customer a package uniformly at random from the remaining packages, he opens the package before delivering it with probability $\frac{1}{2}$. Let $X$ be the number of customers who receive their own packages unopened.

To find $\mathbb{E}[X_i X_j]$, we need to calculate $\mathbb{P}(X_i X_j = 1)$. So, using the chain rule, we have

$$\mathbb{P}(X_i X_j = 1) = \mathbb{P}(X_i = 1)\mathbb{P}(X_j = 1 | X_i = 1) = \frac{1}{2n} \cdot \frac{1}{2(n-1)}$$

since if the $i^{\text{th}}$ customer has received their own package, then the $j^{\text{th}}$ customer has $n-1$ choices left. Hence,

$$\mathbb{E}[X^2] = n \cdot \frac{1}{2n} + n \cdot (n-1) \cdot \frac{1}{2n} \cdot \frac{1}{2(n-1)} = \frac{3}{4}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{3}{4} - \left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

# Task 12

Jonathan and Yiming are playing a card game. The cards have not yet been dealt from the deck to their hands. This deck has $k > 2$ cards, and each card has a real number written on it. In this deck, the sum of the card values is $0$, and that the sum of squares of the values of the cards is $1$. Specifically, if the card values are $c_1, c_2, \ldots, c_k$, then we have $\sum_{i=1}^{k} c_i = 0$ and $\sum_{i=1}^{k} c_i^2 = 1$.

The cards are then going to be dealt randomly in the following fashion: for each card in the deck, a fair coin is flipped. If the coin lands heads, then the card goes to Yiming, and if the coin lands tails, the card goes to Jonathan. Note that it is possible for either player to end up with no cards/all the cards.

Calculate $\mathbb{E}[S]$ and $\mathrm{Var}(S)$, where $S$ is the sum of value of cards in Yiming's hand (where an empty hand corresponds to a sum of $0$). The answer should not include a summation.

# Task 12 - Sol.

Jonathan and Yiming are playing a card game. The cards have not yet been dealt from the deck to their hands. This deck has $k > 2$ cards, and each card has a real number written on it. In this deck, the sum of the card values is 0, and that the sum of squares of the values of the cards is 1. Specifically, if the card values are $c_1, c_2, \ldots, c_k$, then we have $\sum_{i=1}^k c_i = 0$ and $\sum_{i=1}^k c_i^2 = 1$.

The cards are then going to be dealt randomly in the following fashion: for each card in the deck, a fair coin is flipped. If the coin lands heads, then the card goes to Yiming, and if the coin lands tails, the card goes to Jonathan. Note that it is possible for either player to end up with no cards/all the cards.

Calculate $\mathbb{E}[S]$ and $\mathrm{Var}(S)$, where $S$ is the sum of value of cards in Yiming's hand (where an empty hand corresponds to a sum of 0). The answer should not include a summation.

Let $I_i$ be the indicator random variable where $I_i = 1$ if the $i^{\text{th}}$ card goes to Yiming, and $I_i = 0$ otherwise. Then, we have $S = \sum_{i=1}^k c_i I_i$ as the value of Yiming's hand. Then, we see that $\mathbb{E}[S] = \sum_{i=1}^k c_i \cdot \mathbb{E}[I_i] = \sum_{i=1}^k c_i \cdot \frac{1}{2} = 0 \cdot \frac{1}{2} = 0$ since the probability of getting either heads or tails is $\frac{1}{2}$, and

$$\begin{aligned}
\mathrm{Var}(S) &= \sum_{i=1}^k \mathrm{Var}(c_i I_i) && \text{[Independence of } I_i\text{]} \\
&= \sum_{i=1}^k c_i^2 \mathrm{Var}(I_i) && \text{[Property of Variance]} \\
&= 1 \cdot \mathrm{Var}(I_i)
\end{aligned}$$

Since we know that $I_i$ is a Bernoulli random variable, then its variance is $\mathrm{Var}(I_i) = p(1-p) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Thus, we see that $\mathrm{Var}(S) = \frac{1}{4}$.

## Task 13

[Credit: Berkeley CS 70] Let $X$ and $Y$ be random variables, each taking values in the set {0,1,2}, with joint distribution

$$\mathbb{P}[X=0,Y=0]=1/3 \qquad \mathbb{P}[X=0,Y=1]=0 \qquad \mathbb{P}[X=0,Y=2]=1/3$$
$$\mathbb{P}[X=1,Y=0]=0 \qquad \mathbb{P}[X=1,Y=1]=1/9 \qquad \mathbb{P}[X=1,Y=2]=0$$
$$\mathbb{P}[X=2,Y=0]=1/9 \qquad \mathbb{P}[X=2,Y=1]=1/9 \qquad \mathbb{P}[X=2,Y=2]=0$$

**a)** What are the marginal distributions of $X$ and $Y$?

**b)** What are $\mathbb{E}[X]$ and $\mathbb{E}[Y]$?

**c)** Let $I$ be the indicator that $X=1$, and $J$ be the indicator that $Y=1$. What are $\mathbb{E}[I]$, $\mathbb{E}[J]$ and $\mathbb{E}[IJ]$?

**d)** In general, let $I_A$ and $I_B$ be the indicators for events $A$ and $B$ in a probability space $(\Omega, \mathbb{P})$. What is $\mathbb{E}[I_A I_B]$, in terms of the probability of some event?

# Task 13a

[Credit: Berkeley CS 70] Let $X$ and $Y$ be random variables, each taking values in the set $\{0,1,2\}$, with joint distribution

$$\mathbb{P}[X=0,Y=0]=1/3 \qquad \mathbb{P}[X=0,Y=1]=0 \qquad \mathbb{P}[X=0,Y=2]=1/3$$
$$\mathbb{P}[X=1,Y=0]=0 \qquad \mathbb{P}[X=1,Y=1]=1/9 \qquad \mathbb{P}[X=1,Y=2]=0$$
$$\mathbb{P}[X=2,Y=0]=1/9 \qquad \mathbb{P}[X=2,Y=1]=1/9 \qquad \mathbb{P}[X=2,Y=2]=0$$

a) What are the marginal distributions of $X$ and $Y$?

By the law of total probability

$$\mathbb{P}[X=0]=\mathbb{P}[X=0,Y=0]+\mathbb{P}[X=0,Y=1]+\mathbb{P}[X=0,Y=2]=1/3+0+1/3=2/3$$

and similarly

$$\mathbb{P}[X=1]=0+1/9+0=1/9$$
$$\mathbb{P}[X=2]=1/9+1/9+0=2/9$$

# Task 13a

[Credit: Berkeley CS 70] Let $X$ and $Y$ be random variables, each taking values in the set {0,1,2}, with joint distribution

$$\mathbb{P}[X=0,Y=0]=1/3 \qquad \mathbb{P}[X=0,Y=1]=0 \qquad \mathbb{P}[X=0,Y=2]=1/3$$
$$\mathbb{P}[X=1,Y=0]=0 \qquad \mathbb{P}[X=1,Y=1]=1/9 \qquad \mathbb{P}[X=1,Y=2]=0$$
$$\mathbb{P}[X=2,Y=0]=1/9 \qquad \mathbb{P}[X=2,Y=1]=1/9 \qquad \mathbb{P}[X=2,Y=2]=0$$

a) What are the marginal distributions of $X$ and $Y$?

As a sanity check, these three numbers are all positive and they add up to $2/3+1/9+2/9=1$ as they should. The same kind of calculation gives

$$\mathbb{P}[Y=0]=1/3+0+1/9=4/9$$
$$\mathbb{P}[Y=1]=0+1/9+1/9=2/9$$
$$\mathbb{P}[Y=2]=1/3.$$

# Task 13b

[Credit: Berkeley CS 70] Let $X$ and $Y$ be random variables, each taking values in the set $\{0,1,2\}$, with joint distribution

$$
\begin{aligned}
\mathbb{P}[X=0,Y=0] &= 1/3 & \mathbb{P}[X=0,Y=1] &= 0 & \mathbb{P}[X=0,Y=2] &= 1/3 \\
\mathbb{P}[X=1,Y=0] &= 0 & \mathbb{P}[X=1,Y=1] &= 1/9 & \mathbb{P}[X=1,Y=2] &= 0 \\
\mathbb{P}[X=2,Y=0] &= 1/9 & \mathbb{P}[X=2,Y=1] &= 1/9 & \mathbb{P}[X=2,Y=2] &= 0
\end{aligned}
$$

b) What are $\mathbb{E}[X]$ and $\mathbb{E}[Y]$?

From the above marginal distributions, we can compute

$$
\begin{aligned}
\mathbb{E}[X] &= 0\mathbb{P}[X=0] + 1\mathbb{P}[X=1] + 2\mathbb{P}[X=2] = 5/9 \\
\mathbb{E}[Y] &= 0\mathbb{P}[Y=0] + 1\mathbb{P}[Y=1] + 2\mathbb{P}[Y=2] = 8/9
\end{aligned}
$$

# Task 13c

[Credit: Berkeley CS 70] Let $X$ and $Y$ be random variables, each taking values in the set {0,1,2}, with joint distribution

$$\mathbb{P}[X=0,Y=0]=1/3 \quad \mathbb{P}[X=0,Y=1]=0 \quad \mathbb{P}[X=0,Y=2]=1/3$$
$$\mathbb{P}[X=1,Y=0]=0 \quad \mathbb{P}[X=1,Y=1]=1/9 \quad \mathbb{P}[X=1,Y=2]=0$$
$$\mathbb{P}[X=2,Y=0]=1/9 \quad \mathbb{P}[X=2,Y=1]=1/9 \quad \mathbb{P}[X=2,Y=2]=0$$

c) Let $I$ be the indicator that $X=1$, and $J$ be the indicator that $Y=1$. What are $\mathbb{E}[I]$, $\mathbb{E}[J]$ and $\mathbb{E}[IJ]$?

We know that taking the expectation of an indicator for some event gives the probability of that event, so

$$\mathbb{E}[I]=\mathbb{P}[X=1]=1/9$$
$$\mathbb{E}[J]=\mathbb{P}[Y=1]=2/9.$$

The random variable $IJ$ is equal to one if $I=1$ and $J=1$, and is zero otherwise. In other words, it is the indicator for the event that $I=1$ and $J=1$:

$$\mathbb{E}[IJ]=\mathbb{P}[I=1,J=1]=1/9.$$

# Task 13d

[Credit: Berkeley CS 70] Let $X$ and $Y$ be random variables, each taking values in the set {0,1,2}, with joint distribution

$$\begin{aligned}
\mathbb{P}[X=0,Y=0]&=1/3 & \mathbb{P}[X=0,Y=1]&=0 & \mathbb{P}[X=0,Y=2]&=1/3\\
\mathbb{P}[X=1,Y=0]&=0 & \mathbb{P}[X=1,Y=1]&=1/9 & \mathbb{P}[X=1,Y=2]&=0\\
\mathbb{P}[X=2,Y=0]&=1/9 & \mathbb{P}[X=2,Y=1]&=1/9 & \mathbb{P}[X=2,Y=2]&=0
\end{aligned}$$

d) In general, let $I_A$ and $I_B$ be the indicators for events $A$ and $B$ in a probability space $(\Omega, \mathbb{P})$. What is $\mathbb{E}[I_A I_B]$, in terms of the probability of some event?

By what we said in the previous part of the solution, $I_A I_B$ is the indicator for the event $A \cap B$, so

$$\mathbb{E}[I_A I_B] = \mathbb{P}[A \cap B].$$

# Task 15

Consider a group of $n$ people consisting of $k > 6$ left-handed people and $n - k$ right-handed people. Suppose we toss a coin that has probability $p$ of coming up heads. If it comes up heads, we select 3 people out of the $n$ people, uniformly at random, to be on a committee. If it comes up tails, we select $k - 3$ people out of the $n$, again uniformly at random, to be on the committee. What is the probability that the committee consists entirely of left-handed people? No need to simplify your answer. Just circle your final answer.

# Task 15 - Solution

Consider a group of $n$ people consisting of $k > 6$ left-handed people and $n-k$ right-handed people. Suppose we toss a coin that has probability $p$ of coming up heads. If it comes up heads, we select 3 people out of the $n$ people, uniformly at random, to be on a committee. If it comes up tails, we select $k-3$ people out of the $n$, again uniformly at random, to be on the committee. What is the probability that the committee consists entirely of left-handed people? No need to simplify your answer. Just circle your final answer.

Use the law of total probability to obtain

$$p \cdot \frac{\binom{k}{3}}{\binom{n}{3}} + (1-p) \cdot \frac{\binom{k}{k-3}}{\binom{n}{k-3}}$$

# Task 16

Suppose that 100 distinct balls are thrown independently and uniformly at random into 100 distinct bins. What is the probability that bin 1 has 5 balls in it given that bin 2 has 3 balls in it? No need to simplify your answer. Just circle your final answer.

# Task 16 - Solution

Suppose that 100 distinct balls are thrown independently and uniformly at random into 100 distinct bins. What is the probability that bin 1 has 5 balls in it given that bin 2 has 3 balls in it? No need to simplify your answer. Just circle your final answer.

$$Pr(\text{ bin 1 has 5 balls} \mid \text{bin 2 has 3 balls}) = \frac{Pr(\text{ bin 1 has 5 balls and bin 2 has 3 balls})}{Pr(\text{ bin 2 has 3 balls})}$$

$$= \frac{\binom{100}{5}\binom{95}{3}\left(\frac{1}{100}\right)^{8}\left(\frac{98}{100}\right)^{92}}{\binom{100}{3}\left(\frac{1}{100}\right)^{3}\left(\frac{99}{100}\right)^{97}}$$

# Task 17

Every minute, a random word generator spits out one word uniformly at random from the 3-word set { I , love, to}. The word spit out is independent of words spit out at other times. If we let the generator run for $n$ minutes, what is the expected number of times that the phrase "I love to love" appears? No need to simplify your answer. Just circle your final answer.

# Task 17 - Solution

Every minute, a random word generator spits out one word uniformly at random from the 3-word set { I , love, to}. The word spit out is independent of words spit out at other times. If we let the generator run for $n$ minutes, what is the expected number of times that the phrase "I love to love" appears? No need to simplify your answer. Just circle your final answer.

Use linearity of expectation, where $X_i$ is an indicator random variable that is 1 if the i-th work is "I", the (i+1)st word is "love", the (i+2)nd word is "to" and the (i+3)rd word is "love". Since the phrase can start at any of positions 1 to $n-3$, and $Pr(X_i = 1) = 1/3^4$, the answer is

$$(n-3) \cdot \left(\frac{1}{3}\right)^4$$

# Task 14

a) Give an example of discrete random variables $X$ and $Y$ with the property that $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$. Specify the joint distribution of $X$ and $Y$.

# Task 14 - Solution

a) Give an example of discrete random variables $X$ and $Y$ with the property that $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$. Specify the joint distribution of $X$ and $Y$.

Let $\mathbb{P}(X = 1) = \frac{1}{2}, \mathbb{P}(X = -1) = \frac{1}{2}$ and $Y \equiv X$. Then, $\mathbb{E}[X] = 1\mathbb{P}(X = 1) - 1\mathbb{P}(X = -1) = \frac{1}{2} - \frac{1}{2} = 0$, and $\mathbb{E}[Y] = \mathbb{E}[X]$. Similarly, since $Y = X$, we have that $\mathbb{E}[XY] = \mathbb{E}[X^2] = 1$ and $\mathbb{E}[X]\mathbb{E}[Y] = 0$.

The joint distribution is defined by $\mathbb{P}(X = 1, Y = 1) = \frac{1}{2}, \mathbb{P}(X = -1, Y = -1) = \frac{1}{2}$, 0 otherwise.

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b) Give an example of discrete random variables $X$ and $Y$ that (i) are *not independent* and (ii) have the propery that $\mathbb{E}[XY] = 0, \mathbb{E}[X] = 0, \mathbb{E}[Y] = 0$. Again, specify the joint distribution of $X$ and $Y$.

One example is given by the joint distribution $\mathbb{P}(X = -1, Y = \frac{1}{3}) = \mathbb{P}(X = 1, Y = \frac{1}{3}) = \mathbb{P}(X = 0, Y = -\frac{2}{3}) = \frac{1}{3}$.

These are not independent because $P(Y = \frac{1}{3}) = \frac{2}{3} \neq 1 = \mathbb{P}(Y = \frac{1}{3}|X = 1)$. However, $\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[XY] = 0$.

# PSet3 Q7: “Conditional probability & probability spaces”

Consider a probability space $(\Omega, \text{Pr}(\cdot))$ and suppose that $F$ is an event in this space where $\text{Pr}(F) > 0$. Verify that $(\Omega, \text{Pr}(\cdot \mid F))$ is a valid probability space, i.e., that it satisfies the following required three axioms:

(a) $\text{Pr}(E \mid F) \geq 0$ for all events $E \subseteq \Omega$.

(b) $\text{Pr}(\Omega \mid F) = 1$.

(c) For any two mutually exclusive events $G$ and $H$ in $\Omega$,

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since both the numerator and denominator are nonnegative.

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## The fundamental properties of set algebra [ edit ]

The binary operations of set union (∪) and intersection (∩) satisfy many identities. Several of these identities or "laws" have well established names.

Commutative property:

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$

Associative property:

- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive property:

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Source: https://en.wikipedia.org/wiki/Algebra_of_sets

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# PSet7 Q3: “Duck Hunt”

Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead and hunters fire at the same time, but each chooses his target at random, independently of the others. If each hunter independently hits his target with probability 0.6, use the law of total expectation to compute the expected number of ducks that are hit. Assume that the number of ducks in a flock is a Poisson random variable with mean 6.

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$$E[X] = \sum_{d=0}^{\infty} E[X|D=d]Pr(D=d)$$

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Now let $H_j$ be the indicator variable for whether the $j^{th}$ duck is hit ($H_j = 1$ if the $j^{th}$ duck is hit and $H_j = 0$ if the $j^{th}$ duck is not hit). Using the linearity of expectations, we then have

$$E[X|D=d] = \sum_{j=1}^{d} E[H_j]$$

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We can then say $E[H_j] = Pr(j \text{ hit}) \cdot 1 + Pr(j \text{ not hit}) \cdot 0 = Pr(j \text{ hit}) = 1-(1-\frac{0.6}{d})^{10}$. This value is the same for any duck, so we have:

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$$E[X|D=d] = \sum_{j=1}^{d} E[H_j] = d\left(1-(1-\frac{0.6}{d})^{10}\right)$$

Probability of the j^th duck getting hit by AT LEAST one hunter

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$$\boxed{E[X] = \sum_{d=0}^{\infty} E[X|D=d]Pr(D=d)}$$

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Since we know that $D$ is a Poisson random variable with mean 6, we can say that:

$$Pr(D=d) = e^{-6}\frac{6^d}{d!}$$

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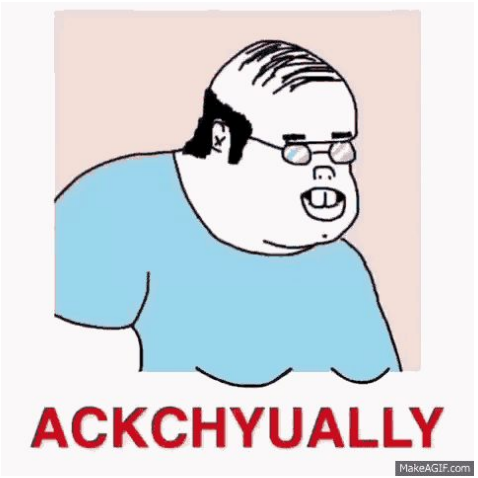
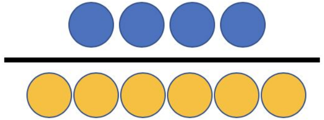
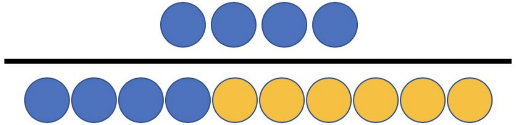
Combining everything into our first expression, we get:

$$E[X] = \sum_{d=0}^{\infty} d\left(1 - (1 - \frac{0.6}{d})^{10}\right)e^{-6}\frac{6^d}{d!}$$

# Extras




Odds = 

Probability = 


Me
Lin Alg, Stats,
Probability Theory
in FreshmanYear
Me
Graduate Level
Machine Learning

# Some Theory Courses

- CSE 421: Algorithms
- CSE 431: Complexity
- CSE 426: Cryptography
- CSE 422: Modern Algorithms

- CSE 446: Machine Learning
- MATH, STAT courses

# THANK YOU!!