CSE 312

Foundations of Computing II

Lecture 11: Zoo of Discrete RVs, part I

Slido.com/3680281

midterm:
2 weeks from today

Agenda

- Recap
- Independent R.V.s and their properties
- Zoo of Discrete RVs
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables

Review Expected Value of a Random Variable

Definition. Given a discrete RV $X : \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of X is

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

or equivalently

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x) \quad = \sum_{x \in \Omega_X} x \cdot p_X(x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

Another Interpretation

"If X is how much you win playing the game in one round. How much would you expect to win, on average, per game, when repeatedly playing?"

Answer: $\mathbb{E}[X]$

Another Interpretation

"If X is how much you win playing the game in one round. How much would you expect to win, <u>on average</u>, per game, when repeatedly playing?"

Answer: $\mathbb{E}[X]$

The Law of Large Numbers*

If $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.) (all have same pmf), then their average value tends to $\mathbb{E}[X]$ with probability 1, i.e.,

$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}[X]\right| \geq \epsilon\right) \to 0 \quad as\ n \to \infty$

Recap Linearity of Expectation

Theorem. For any two random variables X and Y

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

Or, more generally: For any random variables $X_1, \dots, X_n$,

$$\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

Theorem. For any random variables X, and constants a and b

$$\mathbb{E}[aX + b] = a \cdot \mathbb{E}[X] + b.$$

Recap Using LOE to compute complicated expectations

Often boils down to the following three steps:

- Decompose: Finding the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

- LOE: Apply linearity of expectation.

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

- Conquer: Compute the expectation of each X_i

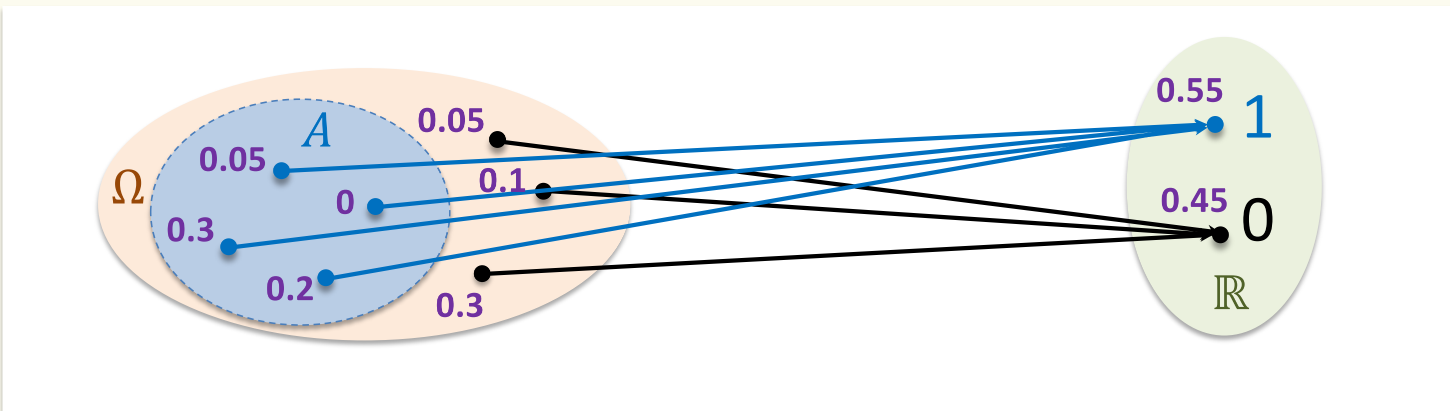
Often, X_i are **indicator** (0/1) random variables.

Recap Indicator random variables – 0/1 valued

For any event A, can define the indicator random variable X_A for A

$$X_A = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{if event } A \text{ does not occur} \end{cases}$$

$$P(X_A = 1) = P(A)$$
$$P(X_A = 0) = 1 - P(A)$$



$$\mathbb{E}[X_A] = P(A) = p$$

Recap Expected Value of $g(X)$ -- LOTUS

$Y = g(X)$

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of $g(X)$ is

$$\mathbb{E}[g(X)] = \sum_{\omega \in \Omega} g\big(X(\omega)\big) \cdot P(\omega)$$

or equivalently

$$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x) \quad = \sum_{x \in \Omega_X} g(x) \cdot p_X(x)$$

Also known as **LOTUS:** "Law of the unconscious statistician

(nothing special going on in the discrete case)

$Y = (X - \mathbb{E}(X))^2$

$Y = g(X)$

$g(x) = (x - \mathbb{E}(X))^2$

Recap Variance – Properties

Definition. The **variance** of a (discrete) RV X is

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_x p_X(x) \cdot (x - \mathbb{E}[X])^2$$

Theorem. $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$\sum_{x \in \Omega_X} x^2 P(X = x)$

Theorem. For any $a, b \in \mathbb{R}$, $\text{Var}(a \cdot X + b) = a^2 \cdot \text{Var}(X)$

consts.

Questions

The **variance** of a (discrete) RV X is

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_x p_X(x) \cdot (x - \mathbb{E}[X])^2.$$

$\text{Var}(X) = E(X^2) - (E(X))^2 \geq 0$

- Can the variance of a random variable be negative? No
- Is Var(X + 5) = Var (X) + 5? $\text{Var}(X+5) = \text{Var}(X)$
- Is it true that if Var(X) = 0, then X is a constant? Yes.
- What is the relationship between $E(X^2)$ and $[E(X)]^2$?

$E(X^2) \geq (E(X))^2$

Recap Variance of Indicator Random Variables

Suppose that X_A is an indicator RV for event A with $P(A) = p$ so

$$\mathbb{E}[X_A] = P(A) = p$$

Since X_A only takes on values 0 and 1, we always have $X_A^2 = X_A$ so

$$\mathrm{Var}(X_A) = \underbrace{\mathbb{E}[X_A^2]}_{p} - \underbrace{\mathbb{E}[X_A]^2}_{p^2} = \mathbb{E}[X_A] - \mathbb{E}[X_A]^2 = p - p^2 = p(1-p)$$

X_A^2 is ind where takes 1 with prob p

$Z = X + Y$

Recap In General, $\mathrm{Var}(X+Y) \neq \mathrm{Var}(X) + \mathrm{Var}(Y)$

Proof by counter-example:

Recall glued coins

- Let X_1 be a r.v. that indicates if the first coin comes up heads.
- Let X_2 be a r.v. that indicates if the second coin comes up heads.

- Outcomes are HT and TH, each with probability 0.5
- Therefore, X_1 and X_2 are indicator random variables with probability 0.5 of being 1.
- Therefore, they both have expectation 0.5 and variance 0.25.
- Thus $Var(X_1) + Var(X_2) = 0.5$
- On the other hand, $X_1 + X_2$ counts the number of heads in the outcome, which is always 1. Therefore $Var(X_1 + X_2) = 0$

$\{X=x\}$ $\{Y=y\}$ indep $\forall$ x, y

Recap Random Variables and Independence

Comma is shorthand for AND

Definition. Two random variables X, Y are **(mutually) independent** if for all $x, y,$

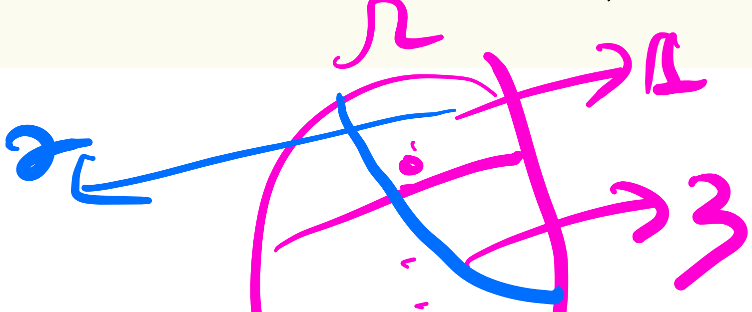
$$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$$

Intuition: Knowing X doesn't help you guess Y and vice versa

Definition. The random variables $X_1, \ldots, X_n$ are **(mutually) independent** if for all $x_1, \ldots, x_n,$

$$P(X_1 = x_1, \ldots, X_n = x_n) = P(X_1 = x_1) \cdots P(X_n = x_n)$$

Note: No need to check for all subsets, but need to check for all values!



Important Facts about Independent Random Variables

Theorem. If X, Y independent, $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Theorem. If X, Y independent, $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$

Corollary. If $X_1, X_2, \dots, X_n$ mutually independent,

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i}^{n} \text{Var}(X_i)$$

Proof not covered

Theorem. If X, Y independent, $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Proof

Let $x_i, y_i, i = 1, 2, \ldots$ be the possible values of X, Y.

$$\mathbb{E}[X \cdot Y] = \sum_i \sum_j x_i \cdot y_j \cdot P(X = x_i \wedge Y = y_j)$$

independence

$$= \sum_i \sum_j x_i \cdot y_i \cdot P(X = x_i) \cdot P(Y = y_j)$$

$$= \sum_i x_i \cdot P(X = x_i) \cdot \left(\sum_j y_j \cdot P(Y = y_j)\right)$$

$$= \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

Note: *NOT* true in general; see earlier example $\mathbb{E}[X^2] \neq \mathbb{E}[X]^2$

Proof not covered

Theorem. If X, Y independent, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Proof

$$Var(X+Y)$$
$$= \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2$$
$$= \mathbb{E}[X^2 + 2XY + Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2$$
$$= \mathbb{E}[X^2] + 2\,\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mathbb{E}[X]^2 + 2\,\mathbb{E}[X]\,\mathbb{E}[Y] + \mathbb{E}[Y]^2)$$
$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2 + \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 + 2\,\mathbb{E}[XY] - 2\,\mathbb{E}[X]\,\mathbb{E}[Y]$$
$$= Var(X) + Var(Y) + 2\,\mathbb{E}[XY] - 2\,\mathbb{E}[X]\,\mathbb{E}[Y]$$
$$= Var(X) + Var(Y)$$

linearity

equal by independence

Example – Coin Tosses

We flip n *independent* coins, each one heads with probability p

- $X_i = \begin{cases} 1, & i^{\text{th}} \text{ outcome is heads} \\ 0, & i^{\text{th}} \text{ outcome is tails.} \end{cases}$ coin toss
- $Z =$ number of heads

Fact. $Z = \sum_{i=1}^{n} X_i$

$P(X_i = 1) = p$
$P(X_i = 0) = 1 - p$
$\mathbb{E}(X_i) = p$

$P(Z = k) = \binom{n}{k} p^k (1-p)^{n-k}$

By LOE $\mathbb{E}[Z] = \sum_{i=1}^{n} \mathbb{E}(X_i) = np$

$\mathrm{Var}(Z) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$ (indep of $X_1 \ldots X_n$)

$\mathrm{Var}(X_i) = p(1-p)$
$= \mathbb{E}(X_i^2) - \mathbb{E}(X_i)^2$

$= np(1-p)$

Example – Coin Tosses

We flip n *independent* coins, each one heads with probability p

- $X_i = \begin{cases} 1, & i^{\text{th}} \text{ outcome is heads} \\ 0, & i^{\text{th}} \text{ outcome is tails.} \end{cases}$
- $Z =$ number of heads

Fact. $Z = \sum_{i=1}^{n} X_i$

$P(X_i = 1) = p$
$P(X_i = 0) = 1 - p$
$\mathbb{E}(X_i) = p$

$P(Z = k) = \binom{n}{k} p^k (1-p)^{n-k}$

By LOE $\mathbb{E}[Z] = \sum_{i=1}^{n} \mathbb{E}(X_i) = np$

Note: $X_1, \dots, X_n$ are mutually independent!

$$\Rightarrow \quad \mathrm{Var}(Z) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = n \cdot p(1-p)$$

Note $\mathrm{Var}(X_i) = p(1-p)$

Agenda

- Independent R.V.s and their properties
- Zoo of Discrete RVs, Part I
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables

Motivation for "Named" Random Variables

Random Variables that show up all over the place.

– Easily solve a problem by recognizing it's a special case of one of these random variables.

Each RV introduced today will show:

– A general situation it models
– Its name and parameters
– Its PMF, Expectation, and Variance
– Example scenarios you can use it

Welcome to the Zoo! (Preview) 🦍🐘🦁🐅🦓🐪🦒

$X \sim \text{Unif}(a, b)$

$$P(X = k) = \frac{1}{b - a + 1}$$
$$\mathbb{E}[X] = \frac{a + b}{2}$$
$$\text{Var}(X) = \frac{(b - a)(b - a + 2)}{12}$$

$X \sim \text{Ber}(p)$

$$P(X = 1) = p, P(X = 0) = 1 - p$$
$$\mathbb{E}[X] = p$$
$$\text{Var}(X) = p(1 - p)$$

$X \sim \text{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$
$$\mathbb{E}[X] = np$$
$$\text{Var}(X) = np(1 - p)$$

$X \sim \text{Geo}(p)$

$$P(X = k) = (1 - p)^{k-1} p$$
$$\mathbb{E}[X] = \frac{1}{p}$$
$$\text{Var}(X) = \frac{1 - p}{p^2}$$

$X \sim \text{NegBin}(r, p)$

$$P(X = k) = \binom{k - 1}{r - 1} p^r (1 - p)^{k-r}$$
$$\mathbb{E}[X] = \frac{r}{p}$$
$$\text{Var}(X) = \frac{r(1 - p)}{p^2}$$

$X \sim \text{HypGeo}(N, K, n)$

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$
$$\mathbb{E}[X] = n\frac{K}{N}$$
$$\text{Var}(X) = n\frac{K(N - K)(N - n)}{N^2(N - 1)}$$

Agenda

- Independent R.V.s and their properties
- Zoo of Discrete RVs, Part I
  – Uniform Random Variables
  – Bernoulli Random Variables
  – Binomial Random Variables
  – Geometric Random Variables

Discrete Uniform Random Variables

A discrete random variable X **equally likely** to take any (integer) value between integers a and b (inclusive), is uniform.

Notation:

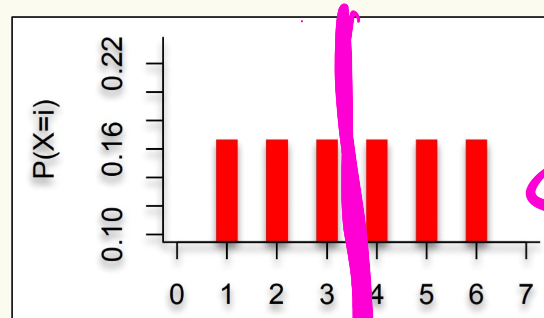
PMF:

Expectation:

Variance:

Example: value shown on one roll of a fair die is $\text{Unif}(1,6)$:

- $P(X = i) = 1/6$
- $\mathbb{E}[X] = 7/2$
- $\text{Var}(X) = 35/12$



X

$Y = 2X + 1$

$E(Y) = 2E(X) + 1$

$Var(Y) = 4\,Var(X)$

Discrete Uniform Random Variables

A discrete random variable X **equally likely** to take any (integer) value between integers a and b (inclusive), is uniform.

Notation: $X \sim \text{Unif}(a, b)$

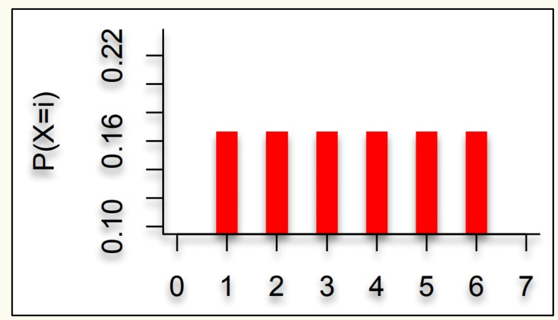
PMF: $\text{P}(X = i) = \frac{1}{b-a+1}$

Expectation: $\mathbb{E}[X] = \frac{a+b}{2}$

Variance: $\text{Var}(X) = \frac{(b-a)(b-a+2)}{12}$

Example: value shown on one roll of a fair die is $\text{Unif}(1,6)$:

- $P(X = i) = 1/6$
- $\mathbb{E}[X] = 7/2$
- $\text{Var}(X) = 35/12$



Agenda

- Independent R.V.s and their properties
- Zoo of Discrete RVs, Part I
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables

Bernoulli Random Variables

~ \sim

A random variable X that takes value 1 ("Success") with probability p, and 0 ("Failure") otherwise. X is called a Bernoulli random variable.

Notation: $X \sim \text{Ber}(p)$

PMF: $P(X = 1) = p,\ P(X = 0) = 1 - p$

Expectation: p

Variance: $p(1-p)$

Poll:
Slido.com/3680281

	Mean	Variance
A.	p	p
B.	p	$1-p$
C.	p	$p(1-p)$
D.	p	p^2

Bernoulli Random Variables

A random variable X that takes value 1 ("Success") with probability p, and 0 ("Failure") otherwise. X is called a Bernoulli random variable.

Notation: $X \sim \text{Ber}(p)$

PMF: $P(X = 1) = p,\ P(X = 0) = 1 - p$

Expectation: $\mathbb{E}[X] = p$ Note: $\mathbb{E}[X^2] = p$

Variance: $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = p - p^2 = p(1 - p)$

Examples:

- Coin flip
- Randomly guessing on a MC test question
- A server in a cluster fails
- Whether or not a share of a particular stock pays off or not
- Any indicator r.v.

Agenda

- Independent R.V.s and their properties
- Zoo of Discrete RVs, Part I
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables

Binomial Random Variables

A discrete random variable $X = \sum_{i=1}^{n} Y_i$ where each $Y_i \sim \text{Ber}(p)$

Counts number of successes in n independent trials, each with probability p of success.

X is a Binomial random variable

Examples:

- # of heads in n indep coin flips
- # of 1s in a randomly generated n bit string
- # of servers that fail in a cluster of n computers
- # of bit errors in file written to disk
- # of elements in a bucket of a large hash table
- # of n different stocks that “pay off”

Poll:
Slido.com/3680281

$P(X = k) =$

A. $p^k(1-p)^{n-k}$
B. np
C. $\binom{n}{k}p^k(1-p)^{n-k}$
D. $\binom{n}{n-k}p^k(1-p)^{n-k}$

elt $h(x)$

m

elts that hash to location 1

$\text{Bin}(n, \frac{1}{m})$

Binomial Random Variables

A discrete random variable $X = \sum_{i=1}^{n} Y_i$ where each $Y_i \sim \text{Ber}(p)$.
Counts number of successes in n independent trials, each with probability p of success.

X is a Binomial random variable

Notation: $X \sim \text{Bin}(n, p)$

PMF: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Expectation:

Variance:

Poll:
Slido.com/3680281

	Mean	Variance
A.	p	p
B.	np	$np(1-p)$
C.	np	np^2
D.	np	n^2p

Binomial Random Variables

A discrete random variable $X = \sum_{i=1}^{n} Y_i$ where each $Y_i \sim \mathrm{Ber}(p)$.

Counts number of successes in n independent trials, each with probability p of success.

X is a Binomial random variable

Notation: $X \sim \mathrm{Bin}(n, p)$

PMF: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Expectation: $\mathbb{E}[X] = np$

Variance: $\mathrm{Var}(X) = np(1-p)$

Mean, Variance of the Binomial

"i.i.d." is a commonly used phrase.
It means "independent & identically distributed"

If $Y_1, Y_2, \dots, Y_n \sim \text{Ber}(p)$ and independent (i.i.d.), then
$X = \sum_{i=1}^{n} Y_i, \quad X \sim \text{Bin}(n, p)$

Claim $\mathbb{E}[X] = np$

LOE

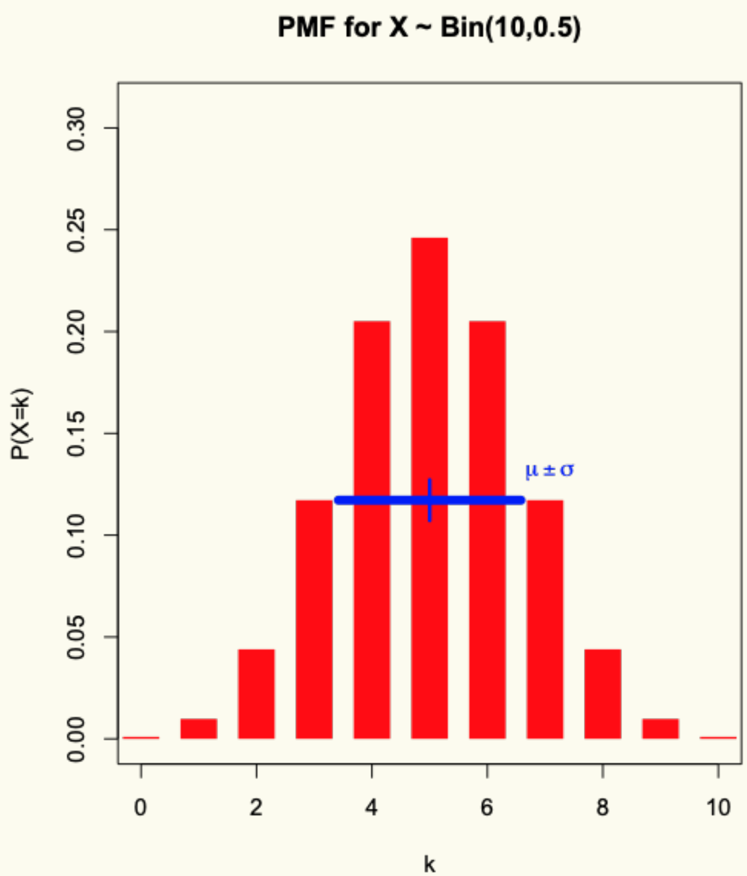
$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = n\mathbb{E}[Y_1] = np$$

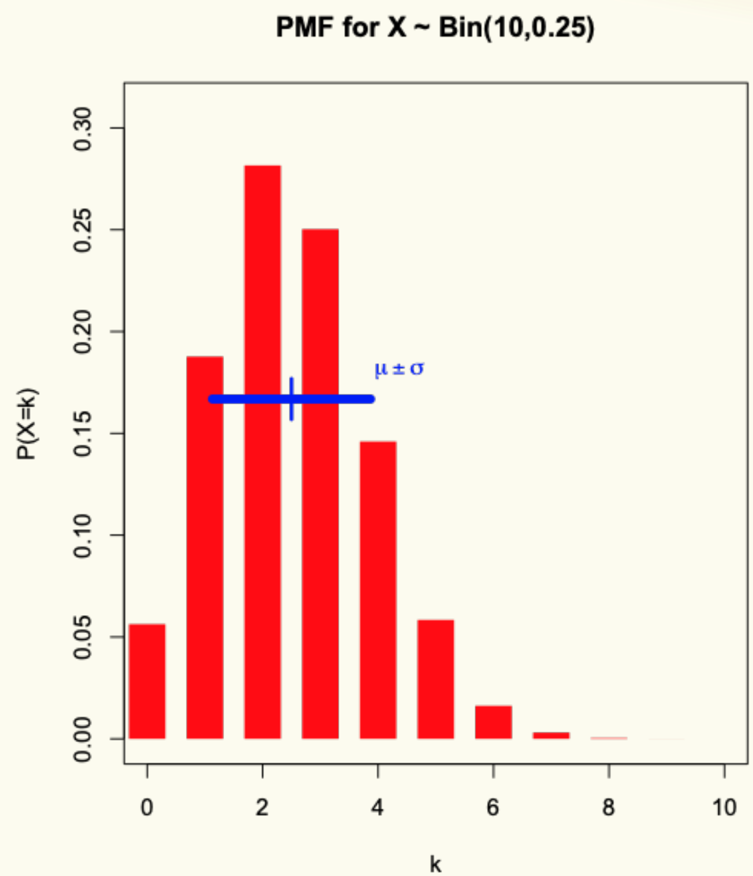
Claim $\text{Var}(X) = np(1-p)$

$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \text{Var}(Y_i) = n\text{Var}(Y_1) = np(1-p)$$

indep

Binomial PMFs





Binomial PMFs