CSE 312

Foundations of Computing II

Lecture 13: Wrap up Poisson r.v.s + Bloom Filters

Anna's office hours on Saturday (tmw) from 2-3pm

(not 3-4pm)

Agenda

- More on Poisson random variables
- An Application: Bloom Filters!

Preview: Poisson

Model: X is # events that occur in an hour

– Expect to see 3 events per hour (but will be random)
– The expected number of events in t hours, is $3t$
– Occurrence of events on disjoint time intervals is independent

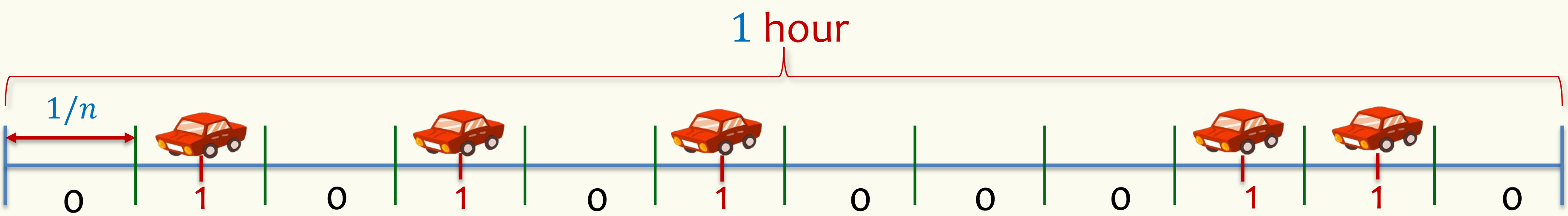
Example – Modelling car arrivals at an intersection

X = # of cars passing through a light in 1 hour

Example – Model the process of cars passing through a light in 1 hour

X = # cars passing through a light in 1 hour.   Disjoint time intervals are independent.

Know: $\mathbb{E}[X] = \lambda$ for some given $\lambda > 0$



Discrete version: n intervals, each of length $1/n$.

In each interval, there is a car with probability $p = \lambda/n$ (assume ≤ 1 car can pass by)

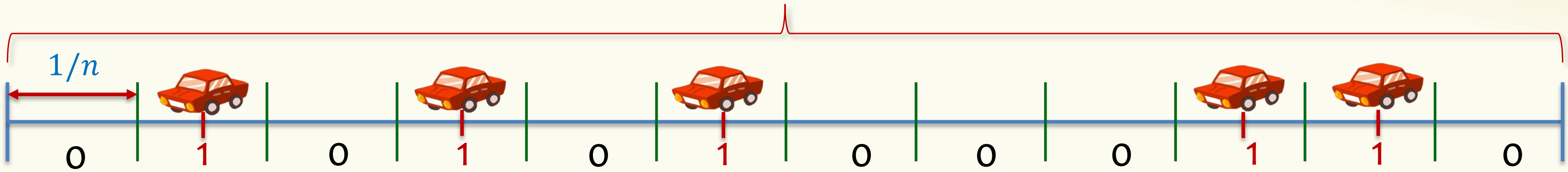
Each interval is Bernoulli: $X_i = 1$ if car in i^{th} interval (0 otherwise). $P(X_i = 1) = \lambda/n$

$X = \sum_{i=1}^{n} X_i$ $X \sim \mathrm{Bin}(n, p)$ $P(X = i) = \binom{n}{i}\left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$

indeed! $\mathbb{E}[X] = pn = \lambda$

Don't like discretization

X is <u>binomial</u> $P(X = i) = \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$



We want now $n \to \infty$

$$P(X = i) = \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i} = \underbrace{\frac{n!}{(n-i)!\, n^i}}_{\to 1} \frac{\lambda^i}{i!} \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{\to e^{-\lambda}} \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-i}}_{\to 1}$$

$$\to P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Poisson Distribution

Siméon Denis Poisson
1781-1840

- Suppose "events" happen, independently, at an *average* rate of λ per unit time.
- Let X be the *actual* number of events happening in a given time unit. Then X is a *Poisson* r.v. *with parameter* λ (denoted $X \sim \mathrm{Poi}(\lambda)$) and has distribution (PMF):

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!} \qquad i = 0, 1, 2, \ldots$$

Several examples of "Poisson processes":
- # of cars passing through a traffic light in 1 hour
- # of requests to web servers in an hour
- # of photons hitting a light detector in a given interval
- # of patients arriving to ER within an hour

Assume fixed average rate

$$\boxed{\begin{gathered} E(X) = \lambda \\ \mathrm{Var}(X) = \lambda \end{gathered}}$$

Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Poisson approximates binomial when:

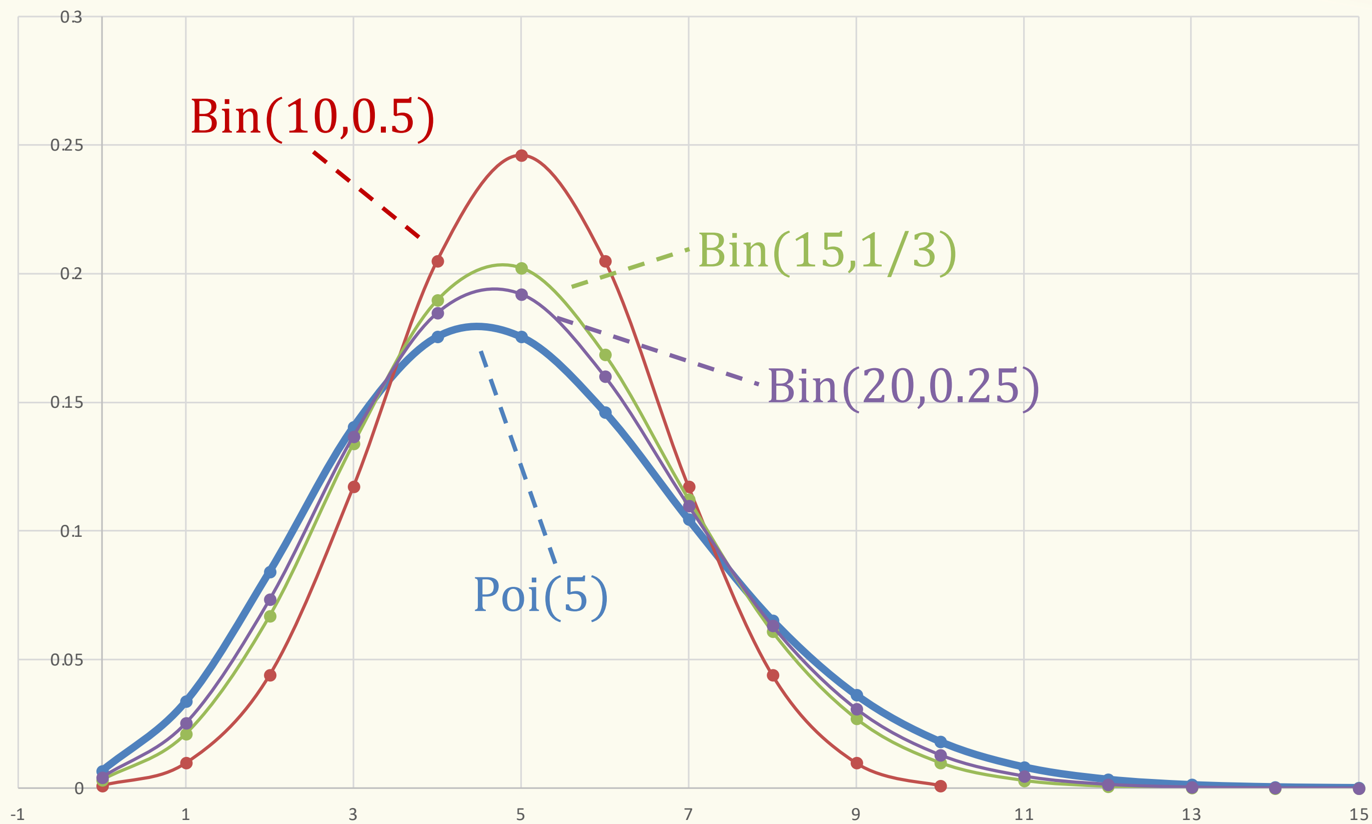
n is very large, p is very small, and $\lambda = np$ is "moderate"

e.g. ($n > 20$ and $p < 0.05$), ($n > 100$ and $p < 0.1$)

Formally, Binomial approaches Poisson in the limit as $n \to \infty$ (equivalently, $p \to 0$) while holding $np = \lambda$

Probability Mass Function – Convergence of Binomials

$\lambda = 5$
$p = \frac{5}{n}$
$n = 10,15,20$



$as\ n \to \infty,\ Binomial(n,\ p = \lambda/n) \to poi(\lambda)$

Sum of Independent Poisson RVs

Let $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.

Let $Z = X + Y$. What kind of random variable is Z ?

Aka what is the "distribution" of Z ?

X, Y indep. $\Downarrow$ $P(X=x, Y=y) = P(X=x)P(Y=y) \quad \forall x,y \in \mathbb{R}$

Intuition first:

- X is measuring number of (type 1) events that happen in, say, an hour if they happen at an average rate of λ_1 per hour.
- Y is measuring number of (type 2) events that happen in, say, an hour if they happen at an average rate of λ_2 per hour.
- Z is measuring total number of events of both types that happen in, say, an hour, if type 1 and type 2 events occur independently.

Sum of Independent Poisson RVs

Theorem. Let $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.
Let $Z = X + Y$. For all $z = 0,1,2,3 \ldots$,

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

indep

More generally, let $X_1 \sim \text{Poi}(\lambda_1), \cdots, X_n \sim \text{Poi}(\lambda_n)$ such that $\lambda = \Sigma_i \lambda_i$.
Let $Z = \Sigma_i X_i$

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

Theorem. Let $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.
Let $Z = X + Y$. For all $z = 0,1,2,3 \ldots,$

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

$z = 0, 1, 2, \ldots$

Proof

$P(Z = z) = \Sigma_{j=0}^{z} P(X = j, Y = z - j)$ Law of total probability

indep $= \sum_{j=0}$

$$(a+b)^z = \sum_{j=0}^{z} \binom{z}{j} a^j b^{z-j}$$

$z!$

$j! \cdot (z-j)!$

Proof

$P(Z = z) = \Sigma_{j=0}^{z} P(X = j, Y = z - j)$ Law of total probability

$= \Sigma_{j=0}^{z} P(X = j)\, P(Y = z - j) = \Sigma_{j=0}^{z}\ e^{-\lambda_1} \cdot \frac{\lambda_1^j}{j!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{z-j}}{z - j!}$ Independence

$= e^{-\lambda_1 - \lambda_2} \left(\Sigma_{j=0}^{z} \cdot \frac{1}{j!\, z - j!} \cdot \lambda_1^j \lambda_2^{z-j} \right)$ $/ \cdot \frac{z!}{z!}$

$= e^{-\lambda} \left(\Sigma_{j=0}^{z} \frac{z!}{j!\, z - j!} \cdot \lambda_1^j \lambda_2^{z-j} \right) \frac{1}{z!}$

$= e^{-\lambda} \cdot (\lambda_1 + \lambda_2)^z \cdot \frac{1}{z!} = e^{-\lambda} \cdot \lambda^z \cdot \frac{1}{z!}$ Binomial Theorem $\lambda = \lambda_1 + \lambda_2$

Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

General principle:

- Events happen at an average rate of λ per time unit
- Number of events happening at a time unit X is distributed according to Poi(λ)

- Poisson approximates Binomial when n is large, p is small, and np is moderate
- Sum of independent Poisson is still a Poisson

Agenda

- Wrap up Poisson random variables
- An Application: Bloom Filters!

Basic Problem

Problem: Store a subset S of a large set U.

Example. $U =$ set of 128 bit strings $\qquad |U| \approx 2^{128}$

$S =$ subset of strings of interest $\qquad |S| \approx 1000$

Two goals:

1. **Very fast** (ideally constant time) answers to queries "Is $x \in S$?" for any $x \in U$.
2. **Minimal storage** requirements.

Naïve Solution I – Constant Time

Idea: Represent S as an array A with 2^{128} entries.

$$A[x] = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

$S = \{0,2, \dots, K\}$

0	1	2	…	K	…		
1	0	1	0	1	…	0	0

Membership test: To check. $x \in S$ just check whether $A[x] = 1$.

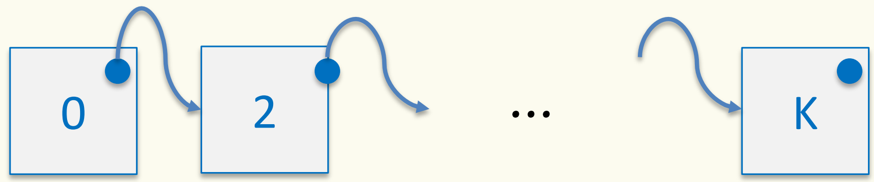
→ **constant time!** 👍 😀

Storage: Require storing 2^{128} bits, even for small S. 👎 😢

Naïve Solution II – Small Storage

Idea: Represent S as a list with $|S|$ entries.

$S = \{0,2,\ldots,K\}$



Storage: Grows with $|S|$ <u>only</u> 👍 😀

Membership test: Check $x \in S$ requires time linear in $|S|$

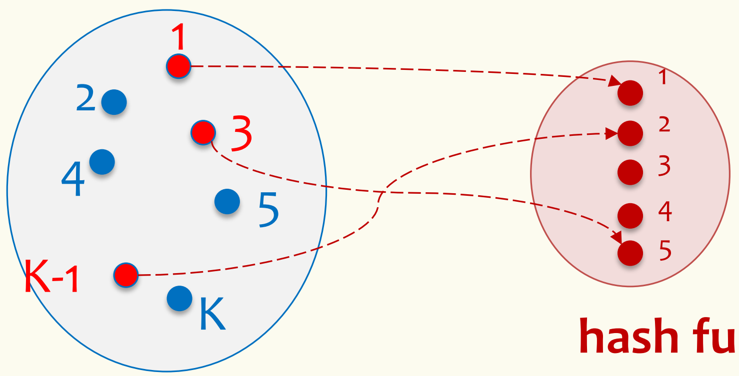
(Can be made logarithmic by using a tree) 👎 😢

Hash Table

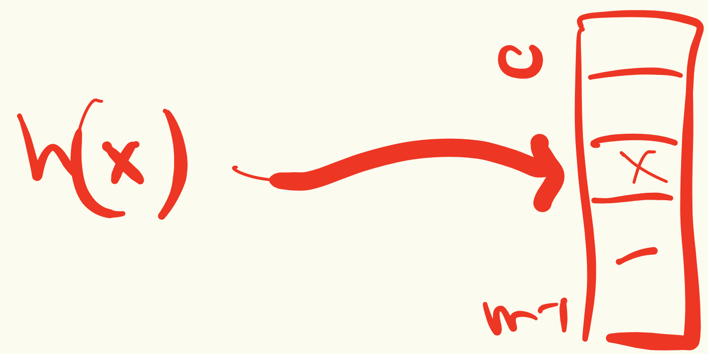
Idea: Map elements in S into an array A of size m using a hash function $\mathbf{h}$

Membership test: To check $x \in S$ just check whether $A[\mathbf{h}(x)] = x$

Storage: m elements (size of array)



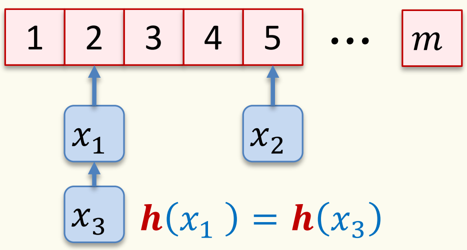
hash function h: $U \to [m]$



Hashing: collisions

Collisions occur when $\boldsymbol{h}(x) = \boldsymbol{h}(y)$ for some distinct $x, y \in S$, i.e., two elements of set map to the same location

- Common solution: chaining – at each location (bucket) in the table, keep linked list of all elements that hash there.



Hash Table

Idea: Map elements in S into an array A of size m using a hash function $\mathbf{h}$

Membership test: To check $x \in S$ just check whether $A[\mathbf{h}(x)] = x$

Storage: m elements (size of array)

Challenge 1: Ensure $\boldsymbol{h}(x) \neq \boldsymbol{h}(y)$ *for most* $x, y \in S$

Challenge 2: Ensure $m = O(|S|)$

Good hash functions to keep collisions low

- The hash function h is good iff it
  - distributes elements uniformly across the m array locations so that
  - pairs of elements are mapped independently

"Universal Hash Functions" – see CSE 332

Hashing: summary

X: # elts that map to location 1 in table

$|S| = m$ elts   table size $= m$

$B(m, \frac{1}{m})$

$E(X) = 1$

Hash Tables

- They store the data itself
- With a good hash function, the data is well distributed in the table and lookup times are small.
- However, they need at least as much space as all the data being stored, i.e., $m = \Omega(|S|)$

In some cases, $|S|$ is huge, or not known a-priori …

Can we do better!?

Bloom Filters
to the rescue

(Named after Burton Howard Bloom)

Bloom Filters – Main points

- Probabilistic data structure.
- Close cousins of hash tables.
  - But: Ridiculously space efficient
- Occasional errors, specifically false positives.

Bloom Filters

- Stores information about a set of elements $S \subseteq U$.
- Supports two operations:
  1. `add`(x) - adds $x \in U$ to the set S
  2. `contains`(x) – ideally: true if $x \in S$, false otherwise

Bloom Filters

- Stores information about a set of elements $S \subseteq U$.
- Supports two operations:
  1. **`add`**(x) - adds $x \in U$ to the set S
  2. **`contains`**(x) – ideally: true if $x \in S$, false otherwise

Instead, relaxed guarantees:
- False $\rightarrow$ **definitely** not in S
- True $\rightarrow$ **possibly** in S

  [i.e. we could have *false positives*]

Bloom Filters – Why Accept False Positives?

when expect most queries to return F

- **Speed** – both `add` and `contains` very very fast.
- **Space** – requires a miniscule amount of space relative to storing all the actual items that have been added.
  - Often just 8 bits per inserted item!
- **Fallback mechanism** – can distinguish false positives from true positives with extra cost
  - Ok if mostly negatives expected + low false positive rate

Bloom Filters: Application

- Google Chrome has a database of malicious URLs, but it takes a long time to query.
- Want an in-browser structure, so needs to be efficient and be space-efficient
- Want it so that can check if a URL is in structure:
  - If return False, then definitely not in the structure (don't need to do expensive database lookup, website is safe)
  - If return True, the URL may or may not be in the structure. Have to perform expensive lookup in this rare case.

Bloom Filters – More Applications

- Any scenario where space and efficiency are important.
- Used a lot in networking
- Internet routers often use Bloom filters to track blocked IP addresses.
- In distributed systems when want to check consistency of data across different locations, might send a Bloom filter rather than the full set of data being stored.
- Google BigTable uses Bloom filters to reduce disk lookups
- And on and on…

Bloom Filters – Ingredients

Basic data structure is a $k \times m$ binary array
"the Bloom filter"

- k rows $t_1, \dots, t_k$, each of size m
- Think of each row as an m-bit vector

k different hash functions $\mathbf{h}_1, \dots, \mathbf{h}_k : U \rightarrow [m]$

Bloom Filters - Initialization

Number of hash functions

Size of array associated to each hash function.

function INITIALIZE(k, m)
  for $i = 1, \ldots, k$: **do**
    t_i = new bit vector of m 0s

for each hash function, initialize an empty bit vector of size m

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function INITIALIZE(k, m)
  for $i = 1, \dots, k$: **do**
    t_i = new bit vector of m 0s

h_1
h_2
h_3

Index →	0	1	2	3	4
t_1	0	0	0	0	0
t_2	0	0	0	0	0
t_3	0	0	0	0	0

Bloom Filters: Add

$$\begin{aligned}&\textbf{function } \text{ADD}(x)\\&\quad \textbf{for } i = 1, \dots, k\textbf{: do}\\&\qquad \underbrace{t_i[h_i(x)]}_{\text{}} = 1\end{aligned}$$

for each hash function $\mathbf{h}_i$

Index into i-th bit-vector, at index produced by hash function and set to 1

$\mathbf{h}_i(x) \rightarrow$ result of hash function $\mathbf{h}_i$ on x

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$\begin{aligned}&\textbf{function } \text{ADD}(x)\\&\quad \textbf{for } i = 1, \dots, k\text{: } \textbf{do}\\&\quad\quad t_i[h_i(x)] = 1\end{aligned}$$

add("thisisavirus.com")

h_1("thisisavirus.com") $\rightarrow$ 2

Index $\rightarrow$	0	1	2	3	4
t_1	0	0	0	0	0
t_2	0	0	0	0	0
t_3	0	0	0	0	0

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$\textbf{function } \text{ADD}(x)$$
$$\quad \textbf{for } i = 1, \dots, k\textbf{: do}$$
$$\quad\quad t_i[h_i(x)] = 1$$

add("thisisavirus.com")

h_1("thisisavirus.com") $\rightarrow$ 2

h_2("thisisavirus.com") $\rightarrow$ 1

Index $\rightarrow$	0	1	2	3	4
t_1	0	0	1	0	0
t_2	0	0	0	0	0
t_3	0	0	0	0	0

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$\begin{aligned}&\textbf{function } \text{ADD}(x)\\&\quad \textbf{for } i = 1, \dots, k\textbf{: do}\\&\quad\quad t_i[h_i(x)] = 1\end{aligned}$$

add("thisisavirus.com")

h_1("thisisavirus.com") → 2

h_2("thisisavirus.com") → 1

h_3("thisisavirus.com") → 4

Index →	0	1	2	3	4
t_1	0	0	1	0	0
t_2	0	1	0	0	0
t_3	0	0	0	0	0

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$\begin{aligned}&\textbf{function } \text{ADD}(x)\\ &\quad \textbf{for } i = 1, \dots, k\textbf{: do}\\ &\qquad t_i[h_i(x)] = 1\end{aligned}$$

add("thisisavirus.com")

h_1("thisisavirus.com") → 2

h_2("thisisavirus.com") → 1

h_3("thisisavirus.com") → 4

Index →	0	1	2	3	4
t_1	0	0	1	0	0
t_2	0	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: Contains

function CONTAINS(x)
  return $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

Returns True if the bit vector t_i for each hash function has bit 1 at index determined by $h_i(x)$,
Returns False otherwise

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function CONTAINS(x)
  return $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

contains("thisisavirus.com")

Index →	0	1	2	3	4
t_1	0	0	1	0	0
t_2	0	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function CONTAINS(x)
  return $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

True

contains("thisisavirus.com")

h_1("thisisavirus.com") → 2

Index →	0	1	2	3	4
t_1	0	0	1	0	0
t_2	0	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function CONTAINS(x)
    return $t_1[h_1(x)] == 1 \land t_2[h_2(x)] == 1 \land \cdots \land t_k[h_k(x)] == 1$

True True

contains("thisisavirus.com")

h_1("thisisavirus.com") → 2

h_2("thisisavirus.com") → 1

Index →	0	1	2	3	4
t_1	0	0	1	0	0
t_2	0	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function CONTAINS(x)
  return $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

True True True

contains("thisisavirus.com")

h_1("thisisavirus.com") $\rightarrow$ 2

h_2("thisisavirus.com") $\rightarrow$ 1

h_3("thisisavirus.com") $\rightarrow$ 4

Index $\rightarrow$	0	1	2	3	4
t_1	0	0	1	0	0
t_2	0	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function CONTAINS(x)
    return $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

True   True   True

contains("thisisavirus.com")

h_1("thisisavirus.com") → 2

h_2("thisisavirus.com") → 1

h_3("thisisavirus.com") → 4

Index →	0	1	2	3	4
t_1	0	0	1	0	0
t_2	0	1	0	0	0
t_3	0	0	0	0	1

Since all conditions satisfied, returns True (correctly)

Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

add("totallynotsuspicious.com")

$$\begin{aligned}&\textbf{function } \textsc{Add}(x)\\&\quad \textbf{for } i = 1, \ldots, k\textbf{: do}\\&\quad\quad t_i[h_i(x)] = 1\end{aligned}$$

Index →	0	1	2	3	4
t_1	0	0	1	0	0
t_2	0	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

add("totallynotsuspicious.com")

h_1("totallynotsuspicious.com") $\rightarrow$ 1

$$\begin{array}{l} \textbf{function } \text{ADD}(x) \\ \quad \textbf{for } i = 1, \dots, k\text{: } \textbf{do} \\ \quad\quad t_i[h_i(x)] = 1 \end{array}$$

Index $\rightarrow$	0	1	2	3	4
t_1	0	0	1	0	0
t_2	0	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$
\begin{aligned}
&\textbf{function } \text{ADD}(x) \\
&\quad \textbf{for } i = 1, \dots, k\textbf{: do} \\
&\qquad t_i[h_i(x)] = 1
\end{aligned}
$$

add("totallynotsuspicious.com")

h_1("totallynotsuspicious.com") → 1

h_2("totallynotsuspicious.com") → 0

Index →	0	1	2	3	4
t_1	0	1	1	0	0
t_2	0	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$\textbf{function } \text{ADD}(x)$$
$$\quad \textbf{for } i = 1, \dots, k\textbf{: do}$$
$$\qquad t_i[h_i(x)] = 1$$

add(“totallynotsuspicious.com”)

h_1(“totallynotsuspicious.com”) → 1

h_2(“totallynotsuspicious.com”) → 0

h_3(“totallynotsuspicious.com”) → 4

Index →	0	1	2	3	4
t_1	0	1	1	0	0
t_2	1	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function ADD(x)
  for $i = 1, \ldots, k$: **do**
    $t_i[h_i(x)] = 1$

add("totallynotsuspicious.com")

h_1("totallynotsuspicious.com") → 1
h_2("totallynotsuspicious.com") → 0
h_3("totallynotsuspicious.com") → 4

Index →	0	1	2	3	4
t_1	0	1	1	0	0
t_2	1	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function CONTAINS(x)
  return $t_1[h_1(x)] == 1 \land t_2[h_2(x)] == 1 \land \cdots \land t_k[h_k(x)] == 1$

contains("verynormalsite.com")

Index →	0	1	2	3	4
t_1	0	1	1	0	0
t_2	1	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function CONTAINS(x)
  return $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

True

contains("verynormalsite.com")

h_1("verynormalsite.com") $\rightarrow$ 2

Index $\rightarrow$	0	1	2	3	4
t_1	0	1	1	0	0
t_2	1	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function CONTAINS(x)
  return $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

True True

contains("verynormalsite.com")

h_1("verynormalsite.com") → 2

h_2("verynormalsite.com") → 0

Index →	0	1	2	3	4
t_1	0	1	1	0	0
t_2	1	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function CONTAINS(x)
  return $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

True   True   True

contains("verynormalsite.com")

h_1("verynormalsite.com") → 2

h_2("verynormalsite.com") → 0

h_3("verynormalsite.com") → 4

Index →	0	1	2	3	4
t_1	0	1	1	0	0
t_2	1	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function CONTAINS(x)
  return $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

True True True

contains("verynormalsite.com")

h_1("verynormalsite.com") → 2

h_2("verynormalsite.com") → 0

h_3("verynormalsite.com") → 4

Index →	0	1	2	3	4
t_1	0	1	1	0	0
t_2	1	1	0	0	0
t_3	0	0	0	0	1

Since all conditions satisfied, returns True (incorrectly)

Bloom Filters – Three operations

- Set up Bloom filter for $S = \emptyset$

function INITIALIZE(k, m)
  for $i = 1, \dots, k$: **do**
    t_i = new bit vector of m 0s

- Update Bloom filter for $S \leftarrow S \cup \{x\}$

function ADD(x)
  for $i = 1, \dots, k$: **do**
    $t_i[h_i(x)] = 1$

- Check if $x \in S$

function CONTAINS(x)
  return $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

What you can't do with Bloom filters

- There is no delete operation
  - Once you have added something to a Bloom filter for S, it stays

- You can't use a Bloom filter to name any element of S

But what you ***can*** do makes them very effective!


Brain Break

Analysis: False positive probability

Question: For an element $x \in U$, what is the probability that `contains`(x) returns true if `add`(x) was never executed before?

Analysis: False positive probability

Question: For an element $x \in U$, what is the probability that `contains`(x) returns true if `add`(x) was never executed before?

Probability over what?! Over the choice of the $\boldsymbol{h}_1, \dots, \boldsymbol{h}_k$

Assumptions for the analysis:

- Each $\mathbf{h}_i(x)$ is uniformly distributed in $[m]$ for all x and i
- Hash function outputs for each $\mathbf{h}_i$ are mutually independent (not just in pairs)
- Different hash functions are independent of each other

$h_i(x) \quad h_j(y)$

False positive probability – Events

Assume we perform **add**$(x_1), \ldots,$**add**(x_n)

+ **contains**(x) for $x \notin \{x_1, \ldots, x_n\}$

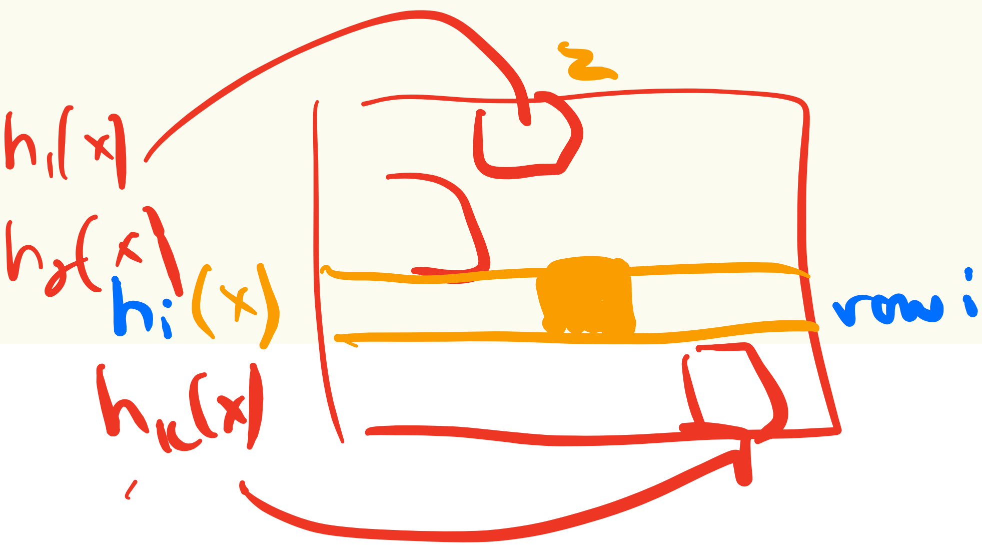
Event E_i holds iff $\mathbf{h}_i(x) \in \{\mathbf{h}_i(x_1), \ldots, \mathbf{h}_i(x_n)\}$

E_i^c $\quad h_i(x) \neq h_i(x_1) \cap \cdots \quad h_i(x) \neq h_i(x_n)$

$z \neq h_i(x_1) \quad z \neq h_i(x_n)$

$$P(\text{false positive}) = P(E_1 \cap E_2 \cap \cdots \cap E_k) = \prod_{i=1}^{k} P(E_i) \quad (1 - P(E_i^c))$$

$\mathbf{h}_1, \ldots, \mathbf{h}_k$ independent



E_i $\qquad$ E_i^c

$h_i(x) = h_i(x_1) \cup h_i(x) = h_i(x_2) \cup h_i(x) = h_i(x_n)$

$h_i(x) \neq h_i(x_1) \cap h_i(x) \neq h_i(x_2) \cap h_i(x) \neq h_i(x_n)$

False positive probability – Events

Event E_i holds iff $\mathbf{h}_i(x) \in \{\mathbf{h}_i(x_1), \dots, \mathbf{h}_i(x_n)\}$

Event E_i^c holds iff $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_1)$ and … and $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_n)$

$$P(E_i^c) = \sum_{z=1}^{m} P(\mathbf{h}_i(x) = z) \cdot P(E_i^c \mid \mathbf{h}_i(x) = z)$$

LTP

False positive probability – Events

Event E_i^c holds iff $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_1)$ and … and $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_n)$

$$P(E_i^c \mid \mathbf{h}_i(x) = z) = P(\mathbf{h}_i(x_1) \neq z, \dots, \mathbf{h}_i(x_n) \neq z \mid \mathbf{h}_i(x) = z)$$

$$= P(\mathbf{h}_i(x_1) \neq z, \dots, \mathbf{h}_i(x_n) \neq z)$$

$$= \prod_{j=1}^{n} P(\mathbf{h}_i(x_j) \neq z)$$

Independence of values of $\boldsymbol{h}_i$ on different inputs

False positive probability – Events

Event E_i^c holds iff $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_1)$ and … and $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_n)$

$$P(E_i^c \mid \mathbf{h}_i(x) = z) = P(\mathbf{h}_i(x_1) \neq z, \dots, \mathbf{h}_i(x_n) \neq z \mid \mathbf{h}_i(x) = z)$$

Independence of values of $\boldsymbol{h}_i$ on different inputs

$$= P(\mathbf{h}_i(x_1) \neq z, \dots, \mathbf{h}_i(x_n) \neq z)$$

$$= \prod_{j=1}^{n} P(\mathbf{h}_i(x_j) \neq z)$$

Outputs of $\boldsymbol{h}_i$ uniformly spread

$$= \prod_{j=1}^{n} \left(1 - \frac{1}{m}\right) = \left(1 - \frac{1}{m}\right)^n$$

$$\Rightarrow P(E_i^c) = \sum_{z=1}^{m} P(\mathbf{h}_i(x) = z) \cdot P(E_i^c \mid \mathbf{h}_i(x) = z) = \left(1 - \frac{1}{m}\right)^n$$

False positive probability – Events

Event E_i holds iff $\mathbf{h}_i(x) \in \{\mathbf{h}_i(x_1), \ldots, \mathbf{h}_i(x_n)\}$

Event E_i^c holds iff $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_1)$ and … and $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_n)$

$$P(E_i^c) = \left(1 - \frac{1}{m}\right)^n$$

$$\Rightarrow \quad \mathrm{FPR} = \prod_{i=1}^{k} \left(1 - P(E_i^c)\right) = \left(1 - \left(1 - \frac{1}{m}\right)^n\right)^k$$

False Positivity Rate – Example

$$\text{FPR} = \left(1 - \left(1 - \frac{1}{m}\right)^{n}\right)^{k}$$

e.g., $n = 5{,}000{,}000$

$k = 30$

$m = 2{,}500{,}000$

FPR = 1.28%

Comparison with Hash Tables - Space

- Google storing 5 million URLs, each URL 40 bytes.
- Bloom filter with $k = 30$ and $m = 2{,}500{,}000$

Hash Table

(optimistic)
$5{,}000{,}000 \times 40B = 200\text{MB}$

Bloom Filter

$2{,}500{,}000 \times 30 = 75{,}000{,}000$ bits

< 10 MB

Time

- Say avg user visits 102,000 URLs in a year, of which 2,000 are malicious.
- 0.5 seconds to do lookup in the database, 1ms for lookup in Bloom filter.
- Suppose the false positive rate is 3%

0.5 seconds DB lookup

false positives

$$\underbrace{1\text{ms}}_{\text{Bloom filter lookup}} + \frac{100000 \times 0.03 \times 500\text{ms} \quad + 2000 \times 500 \text{ ms}}{\underbrace{102000}_{\text{total URLs}}} \approx 25.51\text{ms}$$

malicious URLs

Bloom Filters typical of….

… randomized algorithms and randomized data structures.

- **Simple**
- **Fast**
- **Efficient**
- **Elegant**
- **Useful!**