CSE 312

Foundations of Computing II

Lecture 16: Midterm review

Midterm Monday at 1:30pm (Try to come a few mins early)

- Last name starts with A-F ➔ ECE 125
- Last name starts with G-K ➔ EXED 110
- Last name starts with L-Z ➔ GUG 220 (our regular classroom)

- Can bring one page cheat sheet – must be physical paper. (We will not provide any cheat sheet)

- **Bring your ID with you!**

Student name: ______________ **Student Number:** ______________

CSE 312: Foundations of Computing II - Winter 2024

Midterm

Important: Do not turn the page until instructed to start. In the meantime, read the instructions on this page carefully.

Instructions.

- **Write your name and student number on top of this page.**
- This is a **closed-book exam** with the exception of a one page cheat sheet that you may bring.
- **No electronics are allowed** during the exam (no smart phones, no laptops, no smart watches, no pocket calculators, etc). Before you come into the room where you are taking the test, store them in your bag/backpack and do not take them out until you leave the room after the exam. If we see any such items, we will take your test away.
- Write your solutions in the appropriate spaces (additional empty space is available at the end of the midterm – please add a pointer to where the rest of your solution is.
- You do NOT need to explain your answers and you do NOT need to simplify your answers. In particular, if your answer to a question is correct, you will get full credit *regardless of whether or not you provide any explanation*. If your final answer is wrong though, you may get partial credit if you provide correct partial explanations.
- **Please put a box around each of your final answers**.
- If a problem looks difficult, I recommend moving on to another problem and coming back later.
- You have 50 minutes to complete this midterm.

Good luck!

Probability & Statistics with Applications to Computing
Key Definitions and Theorems

1 Combinatorial Theory

1.1 So You Think You Can Count?

The Sum Rule: If an experiment can either end up being one of N outcomes, or one of M outcomes (where there is no overlap), then the total number of possible outcomes is: $N + M$.

The Product Rule: If an experiment has N_1 outcomes for the first stage, N_2 outcomes for the second stage, $\ldots$, and N_m outcomes for the m^{th} stage, then the total number of outcomes of the experiment is $N_1 \times N_2 \cdot \cdots \cdot N_m = \prod_{i=1}^{m} N_i$.

Permutation: The number of orderings of N **distinct** objects is $N! = N \cdot (N-1) \cdot (N-2) \cdot \ldots 3 \cdot 2 \cdot 1$.

Complementary Counting: Let $\mathcal{U}$ be a (finite) universal set, and S a subset of interest. Then, $\mid S \mid = \mid \mathcal{U} \mid - \mid \mathcal{U} \setminus S \mid$.

1.2 More Counting

***k*-Permutations:** If we want to *pick* (**order matters**) only k out of n distinct objects, the number of ways to do so is:

$$P(n,k) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

***k*-Combinations/Binomial Coefficients:** If we want to *choose* (**order doesn't matter**) only k out of n distinct objects, the number of ways to do so is:

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

Multinomial Coefficients: If we have k distinct types of objects (n total), with n_1 of the first type, n_2 of the second, ..., and n_k of the k-th, then the number of arrangements possible is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \ldots n_k!}$$

Stars and Bars/Divider Method: The number of ways to distribute n indistinguishable balls into k distinguishable bins is

$$\binom{n + (k-1)}{k-1} = \binom{n + (k-1)}{n}$$

1.3 No More Counting Please

Binomial Theorem: Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Principle of Inclusion-Exclusion (PIE):
2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
k events: singles - doubles + triples - quads + ...

Pigeonhole Principle: If there are n pigeons we want to put into k holes (where $n > k$), then at least one pigeonhole must contain at least 2 (or to be precise, $\lceil n/k \rceil$) pigeons.

Combinatorial Proofs: To prove two quantities are equal, you can come up with a combinatorial situation, and show that both in fact count the same thing, and hence must be equal.

2 Discrete Probability

2.1 Discrete Probability

Key Probability Definitions: The **sample space** is the set Ω of all possible outcomes of an experiment. An **event** is any subset $E \subseteq \Omega$. Events E and F are **mutually exclusive** if $E \cap F = \emptyset$.

Axioms of Probability & Consequences:

1. **(Axiom: Nonnegativity)** For any event E, $\mathbb{P}(E) \geq 0$.

2. **(Axiom: Normalization)** $\mathbb{P}(\Omega) = 1$.

3. **(Axiom: Countable Additivity)** If E and F are mutually exclusive, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$.

1. **(Corollary: Complementation)** $\mathbb{P}\left(E^C\right) = 1 - \mathbb{P}(E)$

2. **(Corollary: Monotonicity)** If $E \subseteq F$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$

3. **(Corollary: Inclusion-Exclusion)** $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

Equally Likely Outcomes: If Ω is a sample space such that each of the unique outcome elements in Ω are equally likely, then for any event $E \subseteq \Omega$: $\mathbb{P}(E) = |E|/|\Omega|$.

2.2 Conditional Probability

Conditional Probability: $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

Bayes Theorem: $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$

Partition: Non-empty events $E_1, \ldots, E_n$ **partition** the sample space Ω if they are both:

- **(Exhaustive)** $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^n E_i = \Omega$ (they cover the entire sample space).
- **(Pairwise Mutually Exclusive)** For all $i \neq j$, $E_i \cap E_j = \emptyset$ (none of them overlap)

Note that for any event E, E and E^C always form a partition of Ω.

Law of Total Probability (LTP): If events $E_1, \ldots, E_n$ partition Ω, then for any event F:

$$\mathbb{P}(F) = \sum_{i=1}^{n} \mathbb{P}(F \cap E_n) = \sum_{i=1}^{n} \mathbb{P}(F \mid E_i)\,\mathbb{P}(E_i)$$

Bayes Theorem with LTP: Let events $E_1, \ldots, E_n$ partition the sample space Ω, and let F be another event. Then:

$$\mathbb{P}(E_1 \mid F) = \frac{\mathbb{P}(F \mid E_1)\,\mathbb{P}(E_1)}{\sum_{i=1}^{n} \mathbb{P}(F \mid E_i)\,\mathbb{P}(E_i)}$$

2.3 Independence

Chain Rule: Let $A_1, \ldots, A_n$ be events with nonzero probabilities. Then:

$$\mathbb{P}(A_1, \ldots, A_n) = \mathbb{P}(A_1)\,\mathbb{P}(A_2 \mid A_1)\,\mathbb{P}(A_3 \mid A_1 A_2) \cdots \mathbb{P}(A_n \mid A_1, \ldots, A_{n-1})$$

Independence: A and B are **independent** if any of the following equivalent statements hold:

1. $\mathbb{P}(A \mid B) = \mathbb{P}(A)$
2. $\mathbb{P}(B \mid A) = \mathbb{P}(B)$
3. $\mathbb{P}(A, B) = \mathbb{P}(A)\,\mathbb{P}(B)$

Mutual Independence: We say n events $A_1, A_2, \ldots, A_n$ are **(mutually) independent** if, for *any* subset $I \subseteq [n] = \{1, 2, \ldots, n\}$, we have

$$\mathbb{P}\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} \mathbb{P}(A_i)$$

This equation is actually representing 2^n equations since there are 2^n subsets of $[n]$.

Conditional Independence: A and B are **conditionally independent given an event** C if any of the following equivalent statements hold:

1. $\mathbb{P}(A \mid B, C) = \mathbb{P}(A \mid C)$

2. $\mathbb{P}(B \mid A, C) = \mathbb{P}(B \mid C)$
3. $\mathbb{P}(A, B \mid C) = \mathbb{P}(A \mid C)\,\mathbb{P}(B \mid C)$

3 Discrete Random Variables

3.1 Discrete Random Variables Basics

Random Variable (RV): A random variable (RV) X is a numeric function of the outcome $X : \Omega \to \mathbb{R}$. The set of possible values X can take on is its **range/support**, denoted Ω_X.

If Ω_X is finite or countable infinite (typically integers or a subset), X is a **discrete RV**. Else if Ω_X is uncountably large (the size of real numbers), X is a **continuous RV**.

Probability Mass Function (PMF): For a discrete RV X, assigns probabilities to values in its range. That is $p_X : \Omega_X \to [0, 1]$ where: $p_X(k) = \mathbb{P}(X = k)$.

Expectation: The **expectation** of a discrete RV X is: $\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$.

3.2 More on Expectation

Linearity of Expectation (LoE): For any random variables X, Y (possibly dependent):

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

Law of the Unconscious Statistician (LOTUS): For a discrete RV X and function g, $\mathbb{E}[g(X)] = \sum_{b \in \Omega_X} g(b) \cdot p_X(b)$.

3.3 Variance

Linearity of Expectation with Indicators: If asked only about the expectation of a RV X which is some sort of "count" (and not its PMF), then you may be able to write X as the sum of possibly dependent **indicator** RVs $X_1, \ldots, X_n$, and apply LoE, where for an indicator RV X_i, $\mathbb{E}[X_i] = 1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1)$.

Variance: $\mathsf{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$.

Standard Deviation (SD): $\sigma_X = \sqrt{\mathsf{Var}(X)}$.

Property of Variance: $\mathsf{Var}(aX + b) = a^2 \mathsf{Var}(X)$.

3.4 Zoo of Discrete Random Variables Part I

Independence: Random variables X and Y are **independent**, denoted $X \perp Y$, if for *all* $x \in \Omega_X$ and all $y \in \Omega_Y$: $\mathbb{P}(X = x \cap Y = y) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y)$.

Independent and Identically Distributed (iid): We say $X_1, \ldots, X_n$ are said to be **independent and identically distributed (iid)** if all the X_i's are independent of each other, and have the same distribution (PMF for discrete RVs, or CDF for continuous RVs).

Variance Adds for Independent RVs: If $X \perp Y$, then $\mathsf{Var}(X + Y) = \mathsf{Var}(X) + \mathsf{Var}(Y)$.

Bernoulli Process: A **Bernoulli process** with parameter p is a sequence of independent coin flips $X_1, X_2, X_3, \ldots$ where $\mathbb{P}(\text{head}) = p$. If flip i is heads, then we encode $X_i = 1$; otherwise, $X_i = 0$.

Bernoulli/Indicator Random Variable: $X \sim \text{Bernoulli}(p)$ ($\text{Ber}(p)$ for short) iff X has PMF:

$$p_X(k) = \begin{cases} p, & k = 1 \\ 1 - p, & k = 0 \end{cases}$$

$\mathbb{E}[X] = p$ and $\mathsf{Var}(X) = p(1 - p)$. An example of a Bernoulli/indicator RV is one flip of a coin with $\mathbb{P}(\text{head}) = p$. By a clever trick, we can write

$$p_X(k) = p^k (1 - p)^{1-k}, \quad k = 0, 1$$

Binomial Random Variable: $X \sim \text{Binomial}(n, p)$ ($\mathsf{Bin}(n, p)$ for short) iff X has PMF

$$p_X(k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k \in \Omega_X = \{0, 1, \ldots, n\}$$

$\mathbb{E}[X] = np$ and $\mathsf{Var}(X) = np(1 - p)$. X is the sum of n iid $\text{Ber}(p)$ random variables. An example of a Binomial RV is the number of heads in n independent flips of a coin with $\mathbb{P}(\text{head}) = p$. Note that $\text{Bin}(1, p) \equiv \text{Ber}(p)$. As $n \to \infty$ and $p \to$

0, with $np = \lambda$, then $\text{Bin}(n,p) \to \text{Poi}(\lambda)$. If $X_1, \ldots, X_n$ are independent Binomial RV's, where $X_i \sim \text{Bin}(N_i, p)$, then $X = X_1 + \ldots + X_n \sim \text{Bin}(N_1 + \ldots + N_n, p)$.

3.5 Zoo of Discrete Random Variables Part II

Uniform Random Variable (Discrete): $X \sim \text{Uniform}(a,b)$ ($\text{Unif}(a,b)$ for short), for integers $a \leq b$, iff X has PMF:

$$p_X(k) = \frac{1}{b-a+1}, \quad k \in \Omega_X = \{a, a+1, \ldots, b\}$$

$\mathbb{E}[X] = \frac{a+b}{2}$ and $\text{Var}(X) = \frac{(b-a)(b-a+2)}{12}$. This represents each *integer* in $[a,b]$ to be equally likely. For example, a single roll of a fair die is $\text{Unif}(1,6)$.

Geometric Random Variable: $X \sim \text{Geometric}(p)$ ($\text{Geo}(p)$ for short) iff X has PMF:

$$p_X(k) = (1-p)^{k-1} p, \quad k \in \Omega_X = \{1, 2, 3, \ldots\}$$

$\mathbb{E}[X] = \frac{1}{p}$ and $\text{Var}(X) = \frac{1-p}{p^2}$. An example of a Geometric RV is the number of independent coin flips up to and including the first head, where $\mathbb{P}(\text{head}) = p$.

Negative Binomial Random Variable: $X \sim \text{NegativeBinomial}(r,p)$ ($\text{NegBin}(r,p)$ for short) iff X has PMF:

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k \in \Omega_X = \{r, r+1, r+2, \ldots\}$$

$\mathbb{E}[X] = \frac{r}{p}$ and $\text{Var}(X) = \frac{r(1-p)}{p^2}$. X is the sum of r iid $\text{Geo}(p)$ random variables. An example of a Negative Binomial RV is the number of independent coin flips up to and including the r-th head, where $\mathbb{P}(\text{head}) = p$. If $X_1, \ldots, X_n$ are independent Negative Binomial RV's, where $X_i \sim \text{NegBin}(r_i, p)$, then $X = X_1 + \ldots + X_n \sim \text{NegBin}(r_1 + \ldots + r_n, p)$.

3.6 Zoo of Discrete Random Variables Part III

Poisson Random Variable: $X \sim \text{Poisson}(\lambda)$ ($\text{Poi}(\lambda)$ for short) iff X has PMF:

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \Omega_X = \{0, 1, 2, \ldots\}$$

$\mathbb{E}[X] = \lambda$ and $\text{Var}(X) = \lambda$. An example of a Poisson RV is the number of people born during a particular minute, where λ is the average birth rate per minute. If $X_1, \ldots, X_n$ are independent Poisson RV's, where $X_i \sim \text{Poi}(\lambda_i)$, then $X = X_1 + \ldots + X_n \sim \text{Poi}(\lambda_1 + \ldots + \lambda_n)$.

Hypergeometric Random Variable: $X \sim \text{HyperGeometric}(N, K, n)$ ($\text{HypGeo}(N, K, n)$ for short) iff X has PMF:

$$p_X(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}, \quad k \in \Omega_X = \{\max\{0, n+K-N\}, \ldots, \min\{K, n\}\}$$

$\mathbb{E}[X] = n\frac{K}{N}$ and $\text{Var}(X) = n\frac{K(N-K)(N-n)}{N^2(N-1)}$. This represents the number of successes drawn, when n items are drawn from a bag with N items (K of which are successes, and $N-K$ failures) *without* replacement. If we did this with replacement, then this scenario would be represented as $\text{Bin}\left(n, \frac{K}{N}\right)$.

4 Continuous Random Variables

4.1 Continuous Random Variables Basics

Probability Density Function (PDF): The **probability density function (PDF)** of a continuous RV X is the function $f_X : \mathbb{R} \to \mathbb{R}$, such that the following properties hold:

- $f_X(z) \geq 0$ for all $z \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f_X(t)\, dt = 1$
- $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(w)\, dw$

Cumulative Distribution Function (CDF): The **cumulative distribution function (CDF)** of ANY random variable (discrete or continuous) is defined to be the function $F_X : \mathbb{R} \to \mathbb{R}$ with $F_X(t) = \mathbb{P}(X \leq t)$. If X is a *continuous* RV, we have:

- Suppose that Y is Binomial with parameters 100 and $0.2.$
- Let X be the number of heads if we toss a coin with probability p of coming up Heads independently Y times. What is the probability that $X = k$?

- A professor has a test bank of 20 questions that she will draw on for a particular exam. A particular student knows how to solve 12 of them. The exam she gives is a random subset of 8 of the questions. What is the probability that the student knows how to solve all 8 problems? What is the probability that student knows how to solve exactly 6 of the problems?

True or False

- If X and Y are independent random variables, then so are 5x+3, and 7Y-2.

- Describe the probability mass function of a discrete distribution with mean 10 and variance 9 that takes only 2 distinct values.

- Let Z be a random variable. If $Var(2Z+5) = E(3Z^2) = 12$ and Z is nonnegative, then what is $E(Z)$?

- What is the conditional probability that a random 5-card poker hand is a 4 of a kind (i.e., contains 4 cards of 1 rank and 1 card of a different rank) given that it contains at least one pair?

True or False

- For any events E and F s.t. $Pr(E \mid E \cap F) > 0$, it holds that $Pr(E \mid E \cap F) \leq \Pr(E|F)$

- N voters in a certain country are voting in an election between k candidates: A_1, A_2, … , A_k. Suppose that independently each person votes for candidate A_i with probability p_i, where $\sum_{i=1}^{k} p_i = 1$.
- Let X be the number of votes for either A_1 or A_2 . What distribution from our zoo is X and what are its parameters?

- Same setup. Let X_1 be the number of votes A_1 gets and let X_2 be the number of votes A_2 gets. Are X_1 and X_2 independent?

- Same setup. Let X_i be the number of votes A_i gets and let Z be the number of distinct candidates that get exactly 10 votes. What is $E(Z)$?

A DNA sequence can be thought of as a string made up of 4 bases:

A, T, G, C

Suppose that the DNA sequence is random: the base in each position is selected independently of other positions, and for each particular position, one of the 4 bases is selected such that the letters G and C occur with probability 0.2 each and A and T occur with probability 0.3 each.

In a sequence of length n, what is the expected number of occurrences of the sequence AATGTC?

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

What is $\mathbb{E}[X]$? Use linearity of expectation!

$\mathbf{Pr}(\omega)$	ω	$X(\omega)$
$1/6$	$1,2,3$	3
$1/6$	$1,3,2$	1
$1/6$	$2,1,3$	1
$1/6$	$2,3,1$	0
$1/6$	$3,1,2$	0
$1/6$	$3,2,1$	1

Decompose: What is X_i?

$X_i = 1$ iff i^{th} student gets own HW back

LOE: $X = X_1 + X_2 + \cdots + X_n$

So $\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]$

Conquer: $\mathbb{E}[X_i] = \frac{1}{n}$

Therefore, $\mathbb{E}[X] = n \cdot \frac{1}{n} = 1$

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

What is $\mathbb{E}[X^2]$?

$X_i = 1$ iff i^{th} student gets own HW back

$X = X_1 + X_2 + \cdots + X_n$