CSE 312

Foundations of Computing II

17: Normal Distribution & Central Limit Theorem

Review Continuous RVs

Probability Density Function (PDF).

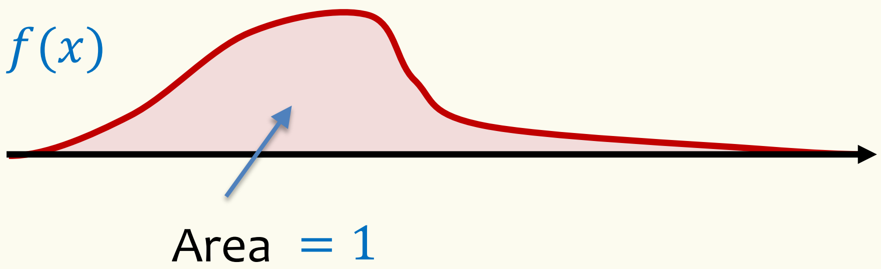
$f: \mathbb{R} \to \mathbb{R}$ s.t.

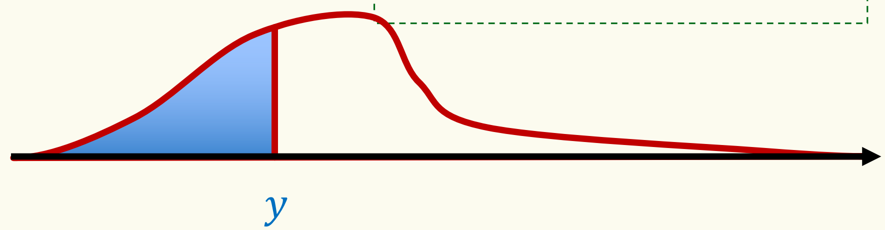
- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x)\, \mathrm{d}x = 1$

Cumulative Distribution Function (CDF).

$$F(y) = \int_{-\infty}^{y} f(x)\, \mathrm{d}x$$

Theorem. $f(x) = \frac{dF(x)}{dx}$





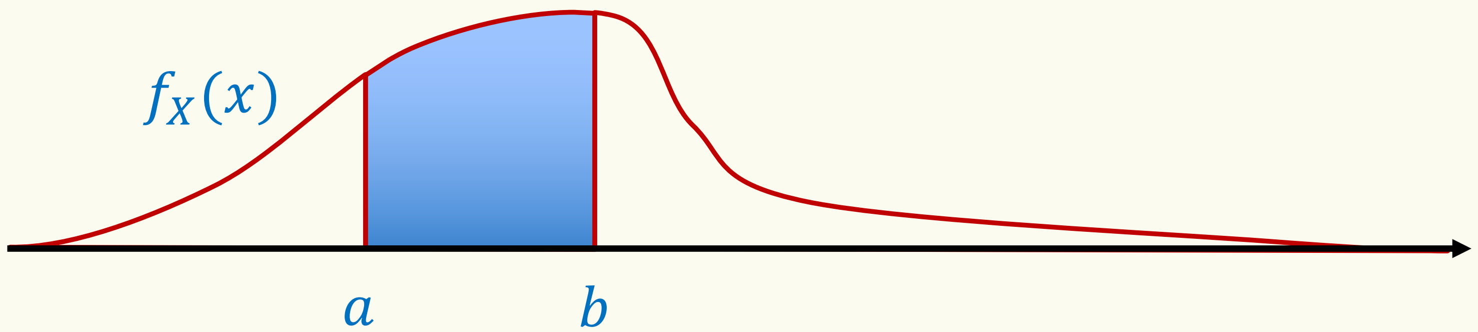
Density ≠ Probability !

$F_X(y) = P(X \leq y)$

but

$\Pr(X \in [x, x+dx]) \approx f(x)\, dx$

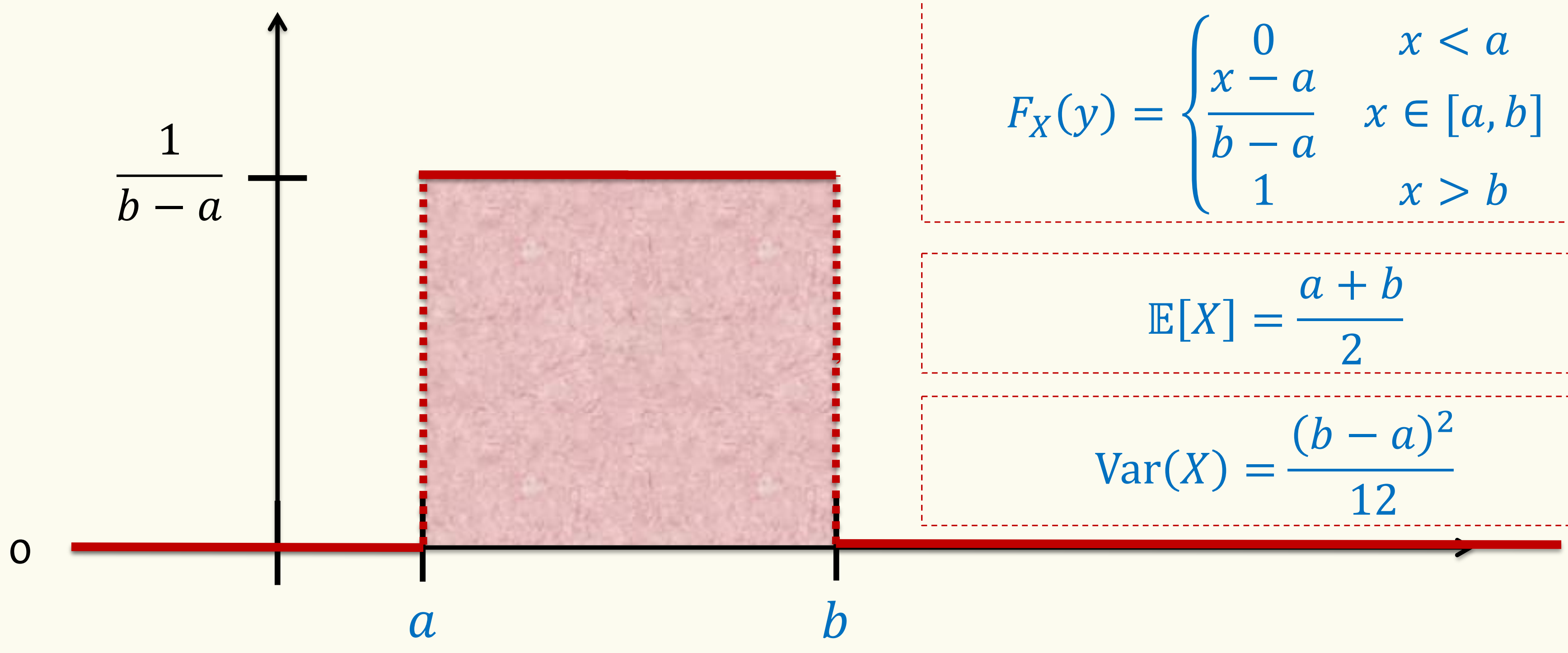
Review Continuous RVs



$$P(X \in [a, b]) = \int_a^b f_X(x)\mathrm{d}x = F_X(b) - F_X(a)$$

Review Uniform Distribution

$X \sim \mathrm{Unif}(a, b)$



$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$$F_X(y) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a,b] \\ 1 & x > b \end{cases}$$

$$\mathbb{E}[X] = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

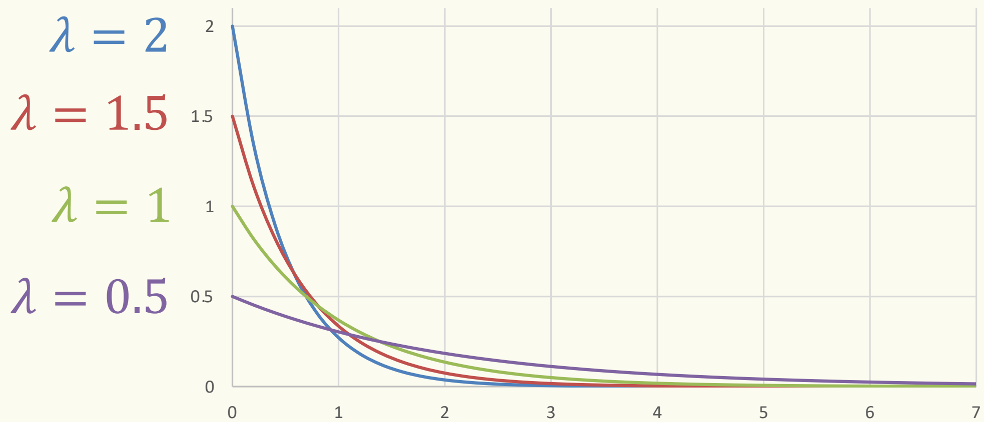
Review Exponential Distribution

Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim \mathrm{Exp}(\lambda)$ and say X that follows the exponential distribution.

CDF: For $y \geq 0$,
$F_X(y) = 1 - e^{-\lambda y}$



Expectation

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$P(X > t) = e^{-t\lambda}$$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

$$= \int_{0}^{+\infty} \lambda e^{-\lambda x} \cdot x \, \mathrm{d}x$$

$$= \left(-(x + \frac{1}{\lambda}) e^{-\lambda x} \right) \Big|_0^{\infty} = \frac{1}{\lambda}$$

Using integration by parts

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$$= \sum_{x \in \Omega_X} g(x)\, p_X(x)$$

$P(X > t) = e^{-t\lambda}$

Exponential Distribution

We write $X \sim \mathrm{Exp}(\lambda)$ and say X that follows the exponential distribution.

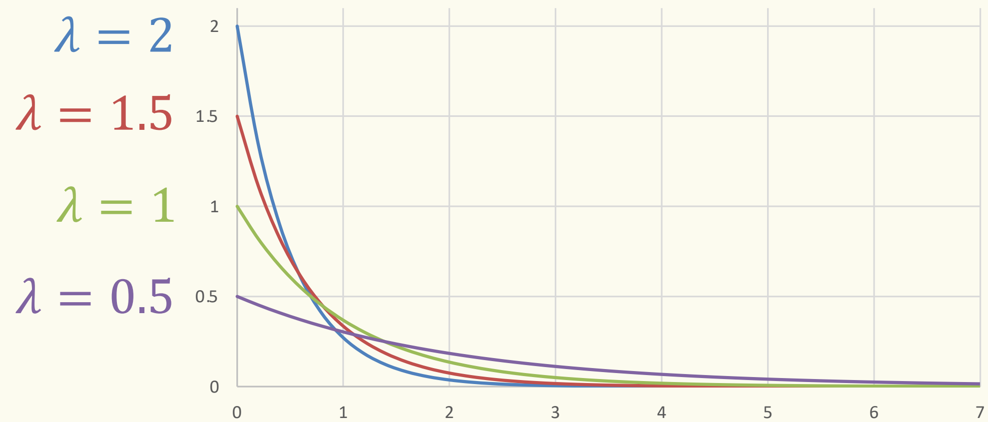
Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

CDF: For $y \geq 0$,
$F_X(y) = 1 - e^{-\lambda y}$

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$



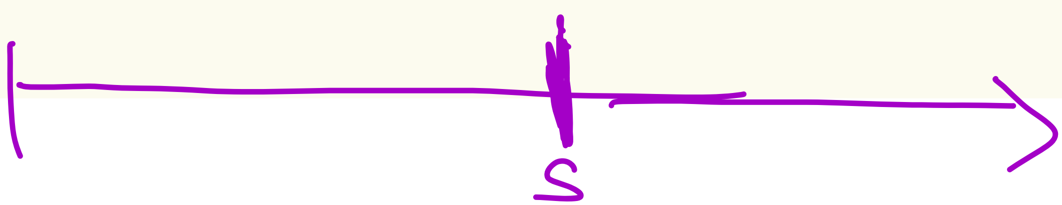
Memorylessness

Definition. A random variable is **memoryless** if for all $s, t > 0$,

$$P(X > s + t \mid X > s) = P(X > t).$$

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

Assuming an exponential distribution, if you've waited s minutes,
The probability of waiting t more is exactly same as when $s = 0$.



Agenda

- Normal Distribution
- Practice with Normals
- Central Limit Theorem (CLT)

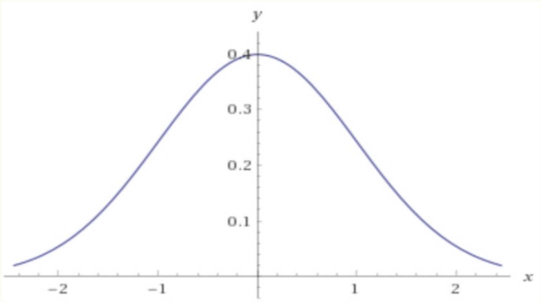
The Normal Distribution

Definition. A **Gaussian (or <u>normal</u>) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma^2 \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Carl Friedrich Gauss

We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$.



$\mathcal{N}(0,1)$.

No closed form expression for CDF…

$$P(X \leq x) = F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

The Normal Distribution. $X \sim \mathcal{N}(\mu, \sigma^2)$

Carl Friedrich Gauss

Definition. A **Gaussian (or normal) random variable** X with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$x = \mu + a$

$x = \mu - a$

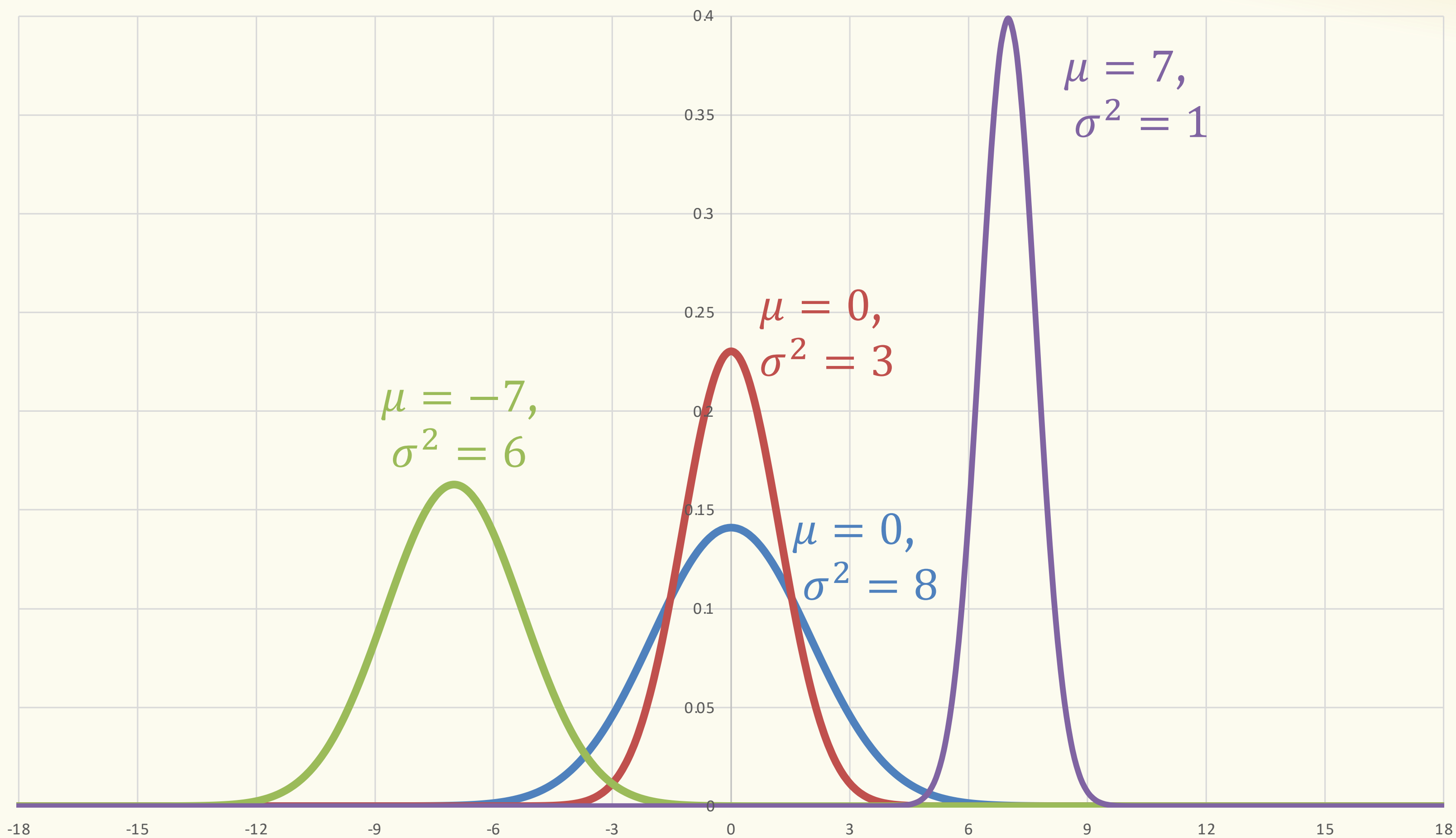
Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}[X] = \mu$, and $\mathrm{Var}(X) = \sigma^2$

Proof of expectation is easy because density curve is symmetric around μ, $f_X(\mu - x) = f_X(\mu + x)$, but proof for variance requires integration of $e^{-x^2/2}$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx \qquad E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

The Normal Distribution

Aka a "Bell Curve" (imprecise name)



Standard normal distribution

Standard (unit) normal = $\mathcal{N}(0,1)$

CDF. $\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \mathrm{d}x$ for $Z \sim \mathcal{N}(0,1)$

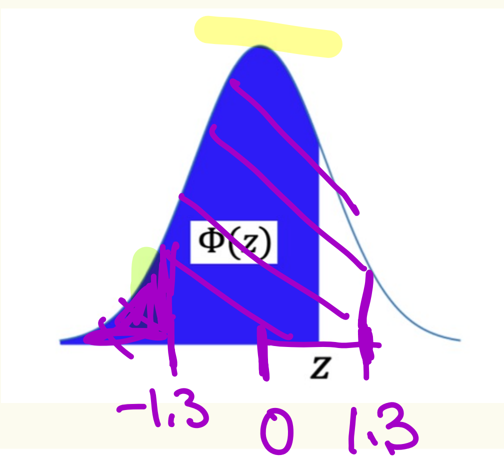
Note: $\Phi(z)$ has no closed form – generally given via tables

$P(Z \leq 1.85) = 0.96784$

Table of Standard Cumulative Normal Density $\mathcal{N}(0, 1)$

$P(Z \leq 0.98) = \Phi(0.98) \approx 0.8365$

$P(Z \leq 1) = \Phi(1.00) \approx 0.84134$

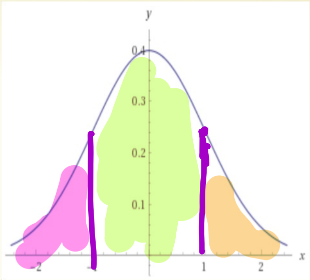


Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

The Standard Normal CDF

$P(Z \leq 1) = \Phi(1.00) \approx 0.84$



What is the probability that a standard Normal is within one standard deviation of its mean?

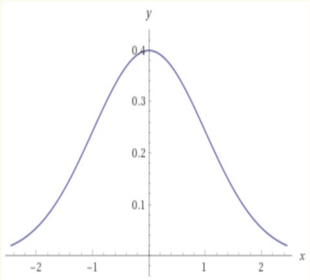
$P(-1 \leq Z \leq 1) =$ $P(Z \leq 1) - \underbrace{P(Z < -1)}_{\text{pink}}$

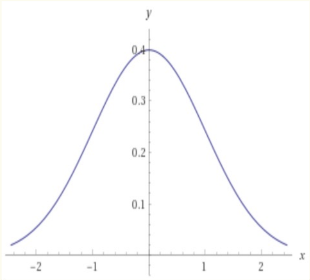
$= P(Z \leq 1) - \underbrace{(1 - P(Z \leq 1))}_{\text{green + pink}}$

orange = pink

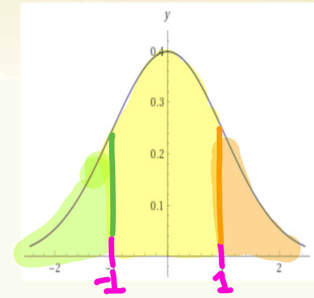
$= 2P(Z \leq 1) - 1$

$= 2 \cdot 0.84... - 1 \approx 0.66$





$P(Z \leq 1) \; = \Phi(1.00) \approx 0.84$



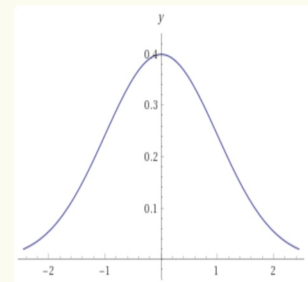
The Standard Normal CDF

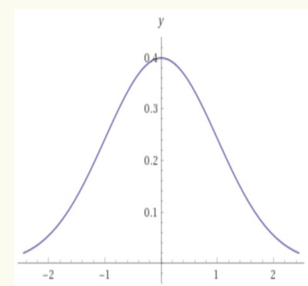
What is the probability that a standard Normal is within one standard deviation of its mean?

$$P(-1 \leq Z \leq 1) = P(Z \leq 1) - P(Z < -1)$$
$$= P(Z \leq 1) - P(Z > 1)$$

symmetry: orange = green

$$= P(Z \leq 1) - (1 - \underbrace{P(Z \leq 1)}_{\text{yellow + green}})$$
$$= 2P(Z \leq 1) - 1$$
$$= 2\Phi(1) - 1$$
$$= 2 \cdot 0.84 - 1 \approx 0.68$$





$\mathrm{Var}(aX) = a^2 \mathrm{Var}(X)$

Important facts about normal distributions

$E(aX) = aE(X)$

$E(X+b) = E(X) + b$

Fact: Normal distributions stay normal under shifting and scaling.

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$

$\frac{X-\mu}{\sigma}$

$E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma} E(X-\mu) = \frac{1}{\sigma}(E(X) - \mu) = 0$

$\mathrm{Var}\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} \mathrm{Var}(X-\mu) = \frac{\mathrm{Var}(X)}{\sigma^2} = 1$

standardizing $\longrightarrow$ mean 0

var 1

Important facts about normal distributions

Fact: Normal distributions stay normal under shifting and scaling.

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$

$E(X) = \mu$

$Var(X) = \sigma^2$

$E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma} E(X-\mu) = \frac{1}{\sigma}[E(X) - \mu] = 0$

$Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} Var(X-\mu) = \frac{1}{\sigma^2} Var(X) = 1$

called "standardizing" the r.v.

– shifting and scaling to make mean 0 and variance 1

Closure of normal distribution – Under Shifting and Scaling

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Mean and variance follow from properties you know! The fact that Y is still normal is not obvious, but not too difficult

Closure of normal distribution – Under Shifting and Scaling

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$

Therefore,

$$F_X(z) = P(X \le z) = P\left(\frac{X-\mu}{\sigma} \le \frac{z-\mu}{\sigma}\right) = \Phi\left(\frac{z-\mu}{\sigma}\right)$$

And can look up the value in the standard normal table.

$\{X \le z\}$ same as $\{X-\mu \le z-\mu\}$

same as $\left\{\frac{X-\mu}{\sigma} \le \frac{z-\mu}{\sigma}\right\}$

Closure of normal distribution – Under Shifting and Scaling

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$$P(X \leq z) = P(X - \mu \leq z - \mu) = Pr\left(\frac{X - \mu}{\sigma} \leq \frac{z - \mu}{\sigma}\right)$$

Agenda

- Normal Distribution
- Practice with Normals
- Central Limit Theorem (CLT)

Example

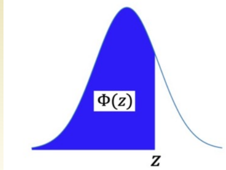
Let $X \sim \mathcal{N}(0.4, 4 = 2^2)$.

$P(X \leq 1.2)$ = $P\left(\frac{X-0.4}{2} \leq \frac{1.2-0.4}{2}\right) = P(Z \leq 0.4)$

$\frac{X-0.4}{2} \to N(0,1)$ $\quad \frac{1.2-0.4}{2} = 0.4$

$= 0.655..$

Table of Standard Cumulative Normal Density



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1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
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3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Example

Let $X \sim \mathcal{N}(0.4, 4 = 2^2)$.

$$P(X \leq 1.2) = P\left(\frac{X - 0.4}{2} \leq \frac{1.2 - 0.4}{2}\right)$$

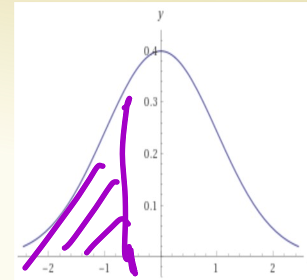
$$= P\left(\frac{X - 0.4}{2} \leq 0.4\right) = \Phi(0.4) \approx 0.6554$$

$\sim \mathcal{N}(0, 1)$

0.1	0.5398	0.5438
0.2	0.5793	0.5832
0.3	0.6179	0.6217
0.4	0.6554	0.6591
0.5	0.6915	0.6950
0.6	0.7257	0.7291
0.7	0.7580	0.7611

Example

$\sigma^2 = 16 \qquad \sigma = 4$



Let $X \sim \mathcal{N}(3, 16)$.

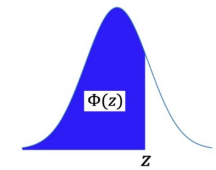
$$P(2 < X < 5) = P\left(\frac{2-3}{4} < \frac{X-3}{4} < \frac{5-3}{4}\right)$$

$$= P\left(-\frac{1}{4} < Z < \frac{1}{2}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{4}\right)$$

$1 - \Phi$

Table of Standard Cumulative Normal Density



Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
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3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Example

Let $X \sim \mathcal{N}(3, 16)$.

$$P(2 < X < 5) = P\left(\frac{2\ -3}{4} < \frac{X-3}{4} < \frac{5\ -3}{4}\right)$$

$$= P\left(-\frac{1}{4} < Z < \frac{1}{2}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{4}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{4}\right)\right) \approx 0.29017$$

Summary so far

- Normal distributions stay normal under shifting and scaling.
- To "standardize" a normal random variable $X \sim \mathcal{N}(\mu, \sigma^2)$, you subtract the mean and divide by the standard deviation, i.e.,

  $$\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

- This allows you to use the standard normal tables (showing $\Phi(z) = P(Z \leq z)$ for $Z \sim \mathcal{N}(0, 1)$) to do calculations for any normal distribution.

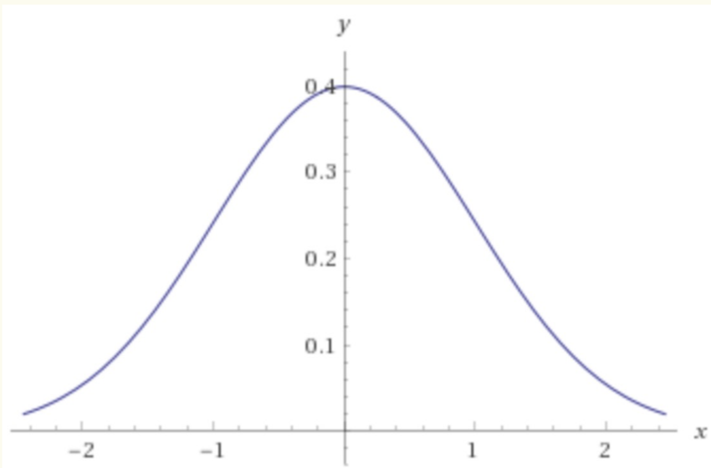
Another important property: closure under addition

Fact. If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $\mathrm{Y} \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent normal RV) then $\mathrm{a}X + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$

Note: The special thing is that **the sum of normal RVs is still a normal RV**. The values of the expectation and variance are **not** surprising.

Why not surprising?

- Linearity of expectation (always true)
- When X and Y are independent, $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$



Normal Distribution

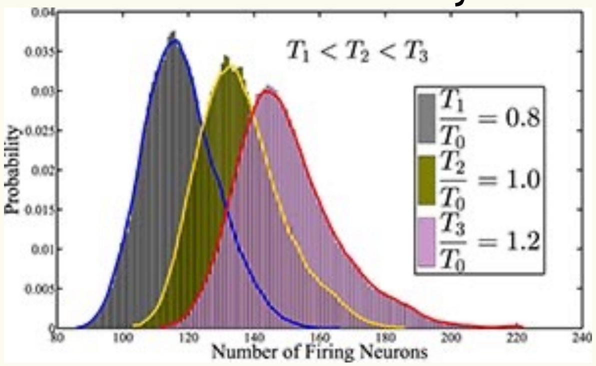
Paranormal Distribution

Agenda

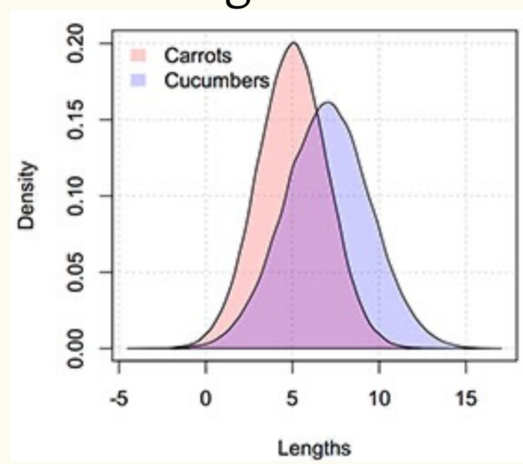
- Normal Distribution
- Practice with Normals
- Central Limit Theorem (CLT)

Normal Distributions EVERYWHERE – why?

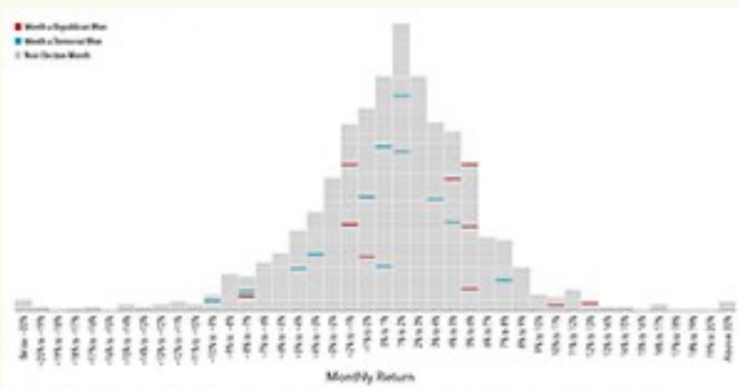
Neuron Activity



Vegetables



S&P 500 Returns after Elections



Examples from:
https://galtonboard.com/probabilityexamplesinlife

Sums of i.i.d. RVs look normal!

i.i.d. = independent and identically distributed

$X_1, \ldots, X_n$ i.i.d. with expectation μ and variance σ^2

Consider $$S_n = X_1 + \cdots + X_n$$

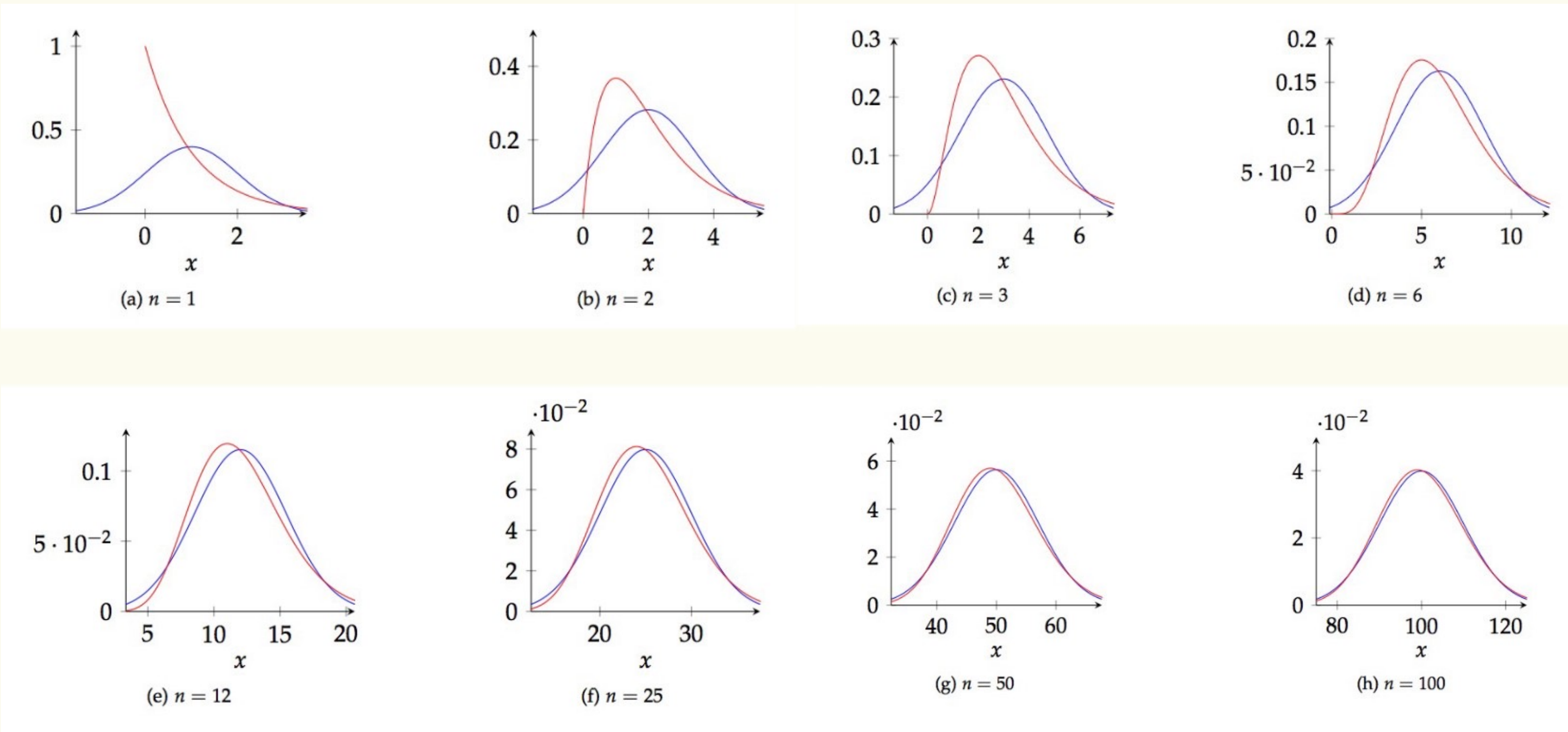
Empirical observation:

S_n looks like a normal RV as n grows.

$$\mathbb{E}[S_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n\mu$$

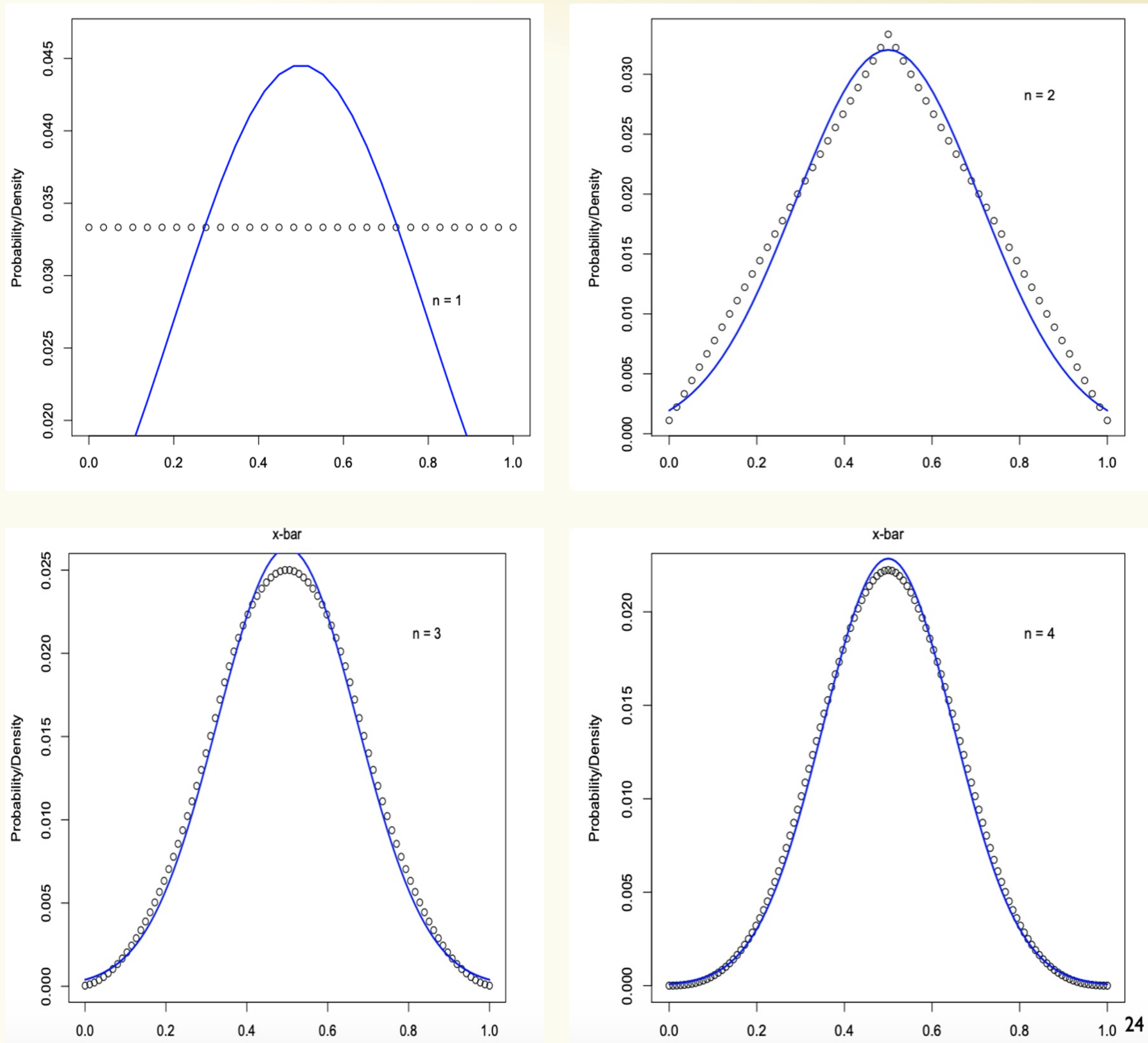
$$\mathrm{Var}(S_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = n\sigma^2$$

Example: Sum of n i.i.d. $\mathrm{Exp}(1)$ random variables

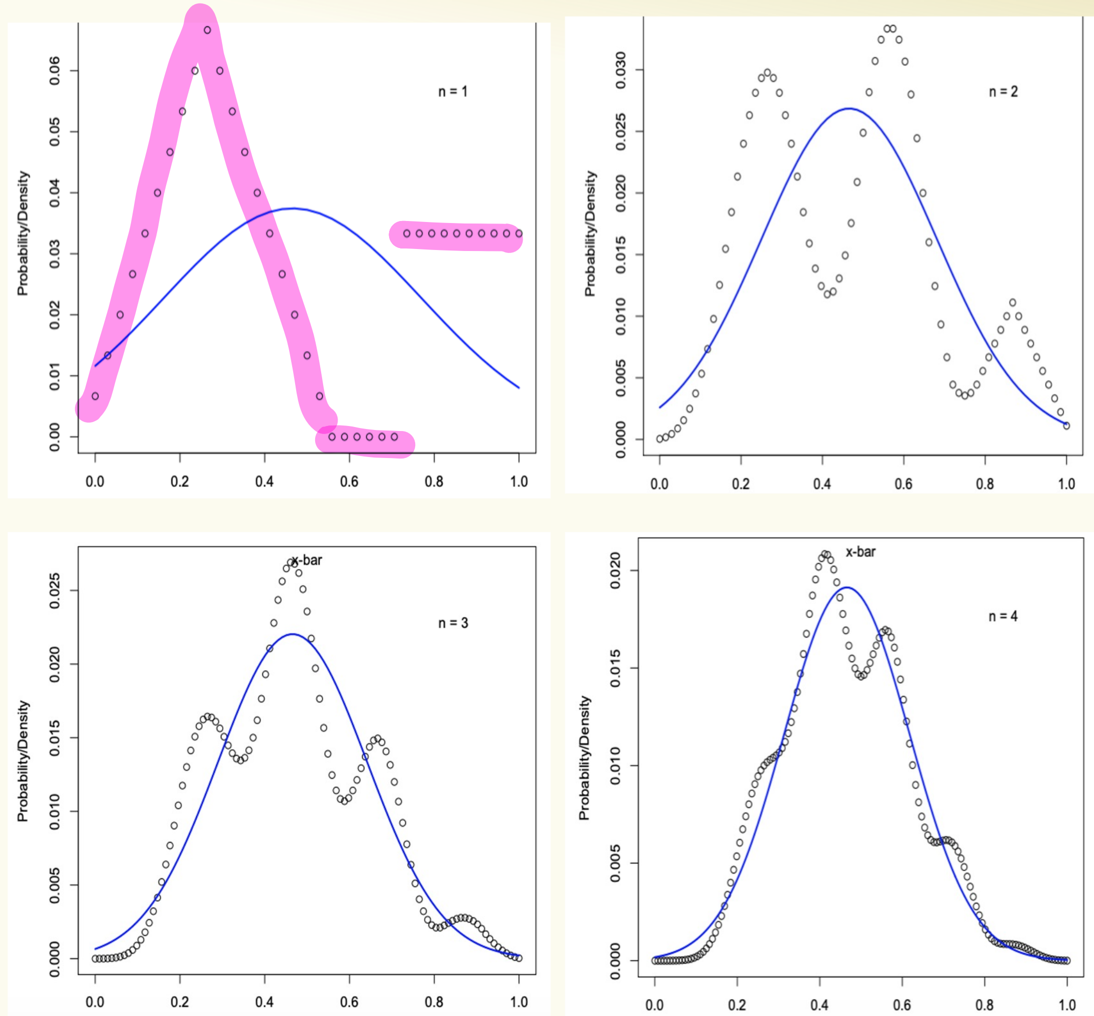


Example: avg of uniform r.v.s

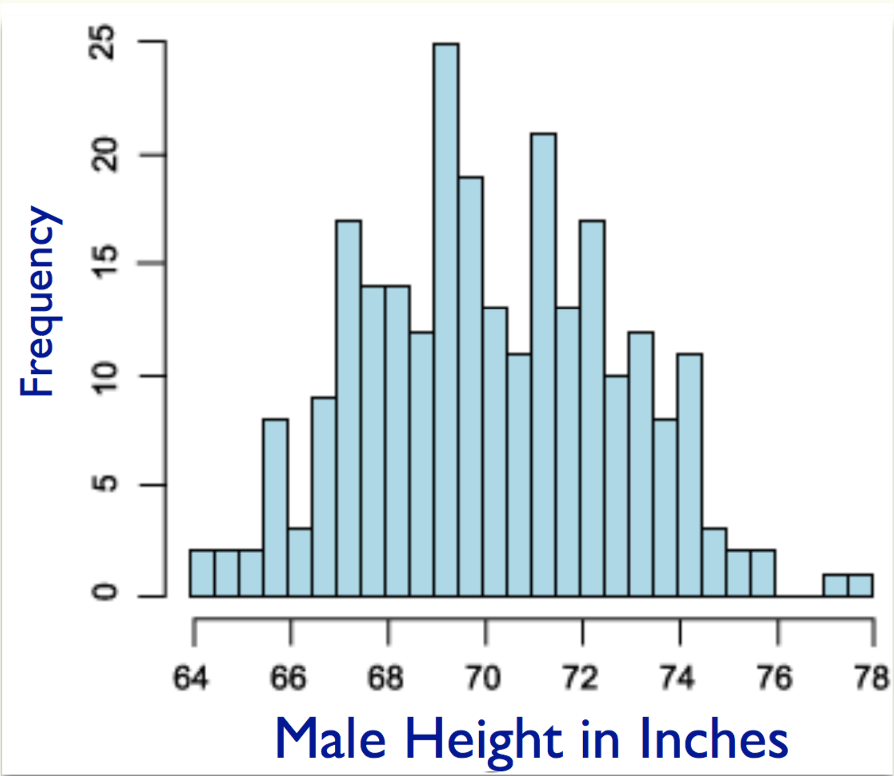
$$\frac{X_1 + X_2 + \dots + X_n}{n}$$



CLT : Avg of some other weird i.i.d. r.v.s



Suppose that what we see in nature results from combining (summing) many independent random observations…



Then distribution might look normal. e.g. Height distribution resembles Gaussian.

R.A.Fisher (1918) observed that the height is likely the outcome of the sum of many independent random parameters, i.e., can written as

$$X = X_1 + \cdots + X_n$$

Sums of i.i.d. RVs

i.i.d. = independent and identically distributed

$X_1, \dots, X_n$ i.i.d. with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$

$$\mathbb{E}[S_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n\mu$$

$$\mathrm{Var}(S_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = n\sigma^2$$

Central Limit Theorem

$X_1, \dots, X_n$ i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$ and

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$\mathbb{E}[S_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n\mu$

$\mathrm{Var}(S_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = n\sigma^2$

$$\mathbb{E}[Y_n] = \frac{1}{\sigma\sqrt{n}}(\mathbb{E}[S_n] - n\mu) = \frac{1}{\sigma\sqrt{n}}(n\mu - n\mu) = 0$$

$$\mathrm{Var}(Y_n) = \frac{1}{\sigma^2 n}\left(\mathrm{Var}(S_n - n\mu)\right) = \frac{\mathrm{Var}(S_n)}{\sigma^2 n} = \frac{\sigma^2 n}{\sigma^2 n} = 1$$

Central Limit Theorem

$X_1, \ldots, X_n$ i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$,

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

Then distribution of $Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$ converges to that of a normal distribution with mean 0 and variance 1 as $n \to \infty$.

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Theorem. (Central Limit Theorem) The CDF of Y_n converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

$$\lim_{n\to\infty} P(Y_n \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \mathrm{d}x$$

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

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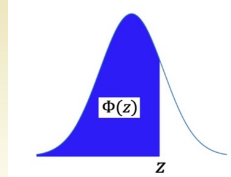
Also stated as:

- $\lim_{n\to\infty} Y_n \to \mathcal{N}(0,1)$
- $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$ for $\mu = \mathbb{E}[X_i]$ and $\sigma^2 = \mathrm{Var}(X_i)$

CLT application

- You buy lightbulbs that burn out according to an exponential distribution with parameter $\lambda = 1.8$ lightbulbs per year.
- You buy a pack of 10 (independent) light bulbs. What is the probability that your 10-pack lasts at least 5 years?

Table of Standard Cumulative Normal Density



Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

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1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
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Summary Central Limit Theorem

$X_1, \dots, X_n$ i.i.d., each with expectation μ and variance σ^2

Define $\boldsymbol{S_n = X_1 + \cdots + X_n}$ and $\bar{\boldsymbol{X}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X_i}$. and $\boldsymbol{Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}}$

mean

variance

CLT:

Summary Central Limit Theorem

$X_1, \dots, X_n$ i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. and $Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$

	S_n	$\bar{X}$	Y_n
mean	$\sum E(X_i) = n\mu$	$\frac{1}{n}\sum_{i=1}^{n} E(X_i) = \mu$	0
variance	$\sum Var(X_i) = n\sigma^2$	$\frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{\sigma^2}{n}$	1
CLT:	$\approx N(n\mu, n\sigma^2)$	$\xrightarrow[n\to\infty]{} N(\mu, \frac{\sigma^2}{n})$	$\xrightarrow[n\to\infty]{} N(0,1)$

Outline of CLT steps – extra step if random variables are discrete.

- Write the event you are interested in, in terms of a sum of i.i.d. random variables.
- Normalize RV to have mean 0 and standard deviation 1.
- Write event in terms of Φ, the CDF of a $\mathcal{N}(0,1)$.
- Look up in table.

Outline of CLT steps – extra step if random variables are discrete.

- Write the event you are interested in, in terms of a sum of i.i.d. random variables.
- Apply continuity correction if RVs are discrete (see tomorrow's section and Tsun Section 5.7.4)
- Normalize RV to have mean 0 and standard deviation 1.
- Write event in terms of Φ, the CDF of a $\mathcal{N}(0,1)$.
- Look up in table.

Example – How Many Standard Deviations Away?

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

$$P(|X-\mu| < k\sigma) = P\left(\frac{|X-\mu|}{\sigma} < k\right) =$$

$$= P\left(-k < \frac{X-\mu}{\sigma} < k\right) = \Phi(k) - \Phi(-k)$$

e.g. $k = 1$: 68%
$k = 2$: 95%
$k = 3$: 99%