CSE 312

Foundations of Computing II

Lecture 18: CLT & Polling

My office hours this weekend will be on Sunday from 4-5pm (instead of Saturday)

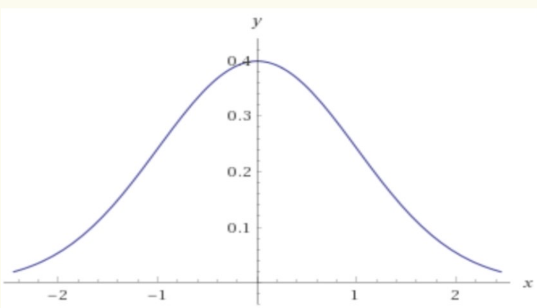
Review The Normal Distribution

Definition. A **Gaussian (or <u>normal</u>) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma^2 \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Carl Friedrich Gauss

We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$.



$\mathcal{N}(0,1)$.

No closed form expression for CDF…

Review The Normal Distribution. $X \sim \mathcal{N}(\mu, \sigma^2)$

Carl Friedrich Gauss

Definition. A **Gaussian (or normal) random variable** X with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}[X] = \mu$, and $\mathrm{Var}(X) = \sigma^2$

Proof of expectation is easy because density curve is symmetric around μ, $f_X(\mu - x) = f_X(\mu + x)$, but proof for variance requires integration of $e^{-x^2/2}$

Review Standard (unit) normal = $\mathcal{N}(0, 1)$

CDF. $\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \mathrm{d}x$ for $Z \sim \mathcal{N}(0, 1)$

Note: $\Phi(z)$ has no closed form – generally given via tables

Review Table of $\Phi(z)$ CDF of Standard Normal

$P(Z \leq 0.98) = \Phi(0.98) \approx 0.8365$

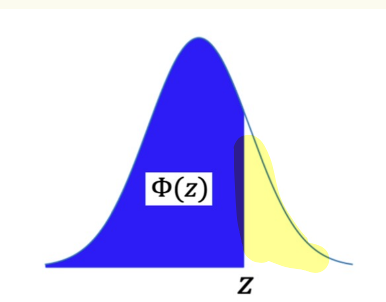
For what a is

$P(Z > a) \leq 0.01$?

$P(Z \leq a) \geq 0.99$

$a \geq 2.33$

$\Rightarrow P(Z \leq a) \geq 0.99$



Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

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1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

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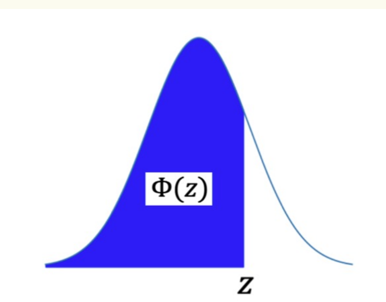

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Review Properties of normal distributions

- Normal distributions stay normal under shifting and scaling.
- To "standardize" a normal random variable $X \sim \mathcal{N}(\mu, \sigma^2)$, you subtract the mean and divide by the standard deviation, i.e.,

  $$\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

- This allows you to use the standard normal tables (showing $\Phi(z) = P(Z \leq z)$ for $Z \sim \mathcal{N}(0, 1)$) to do calculations for any normal distribution.

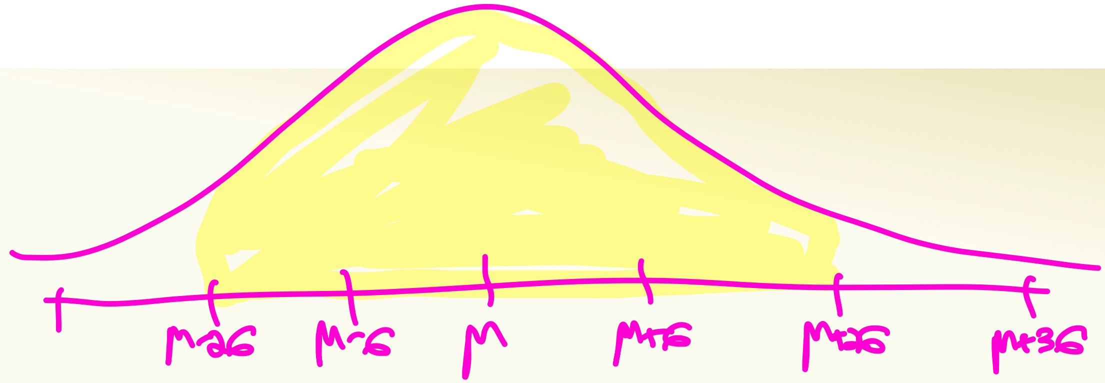
Review Analyzing non-standard normal in terms of $\mathcal{N}(0,1)$

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$

Therefore,

$$F_X(z) = P(X \leq z) = P\left(\frac{X-\mu}{\sigma} \leq \frac{z-\mu}{\sigma}\right) = \Phi\left(\frac{z-\mu}{\sigma}\right)$$

Example



Let $X \sim \mathcal{N}(0, 1)$.

$$P(|X - \mu| < k\sigma) = P\left(\frac{|X - \mu|}{\sigma} < k\right) =$$

$$= P\left(-k < \frac{X - \mu}{\sigma} < k\right) = \Phi(k) - \Phi(-k)$$

e.g. $k = 1$: 68%
$k = 2$: 95%
$k = 3$: 99%

Review closure under addition

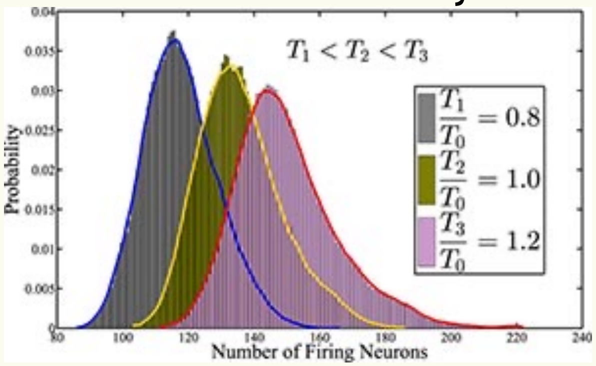
Fact. If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $\mathrm{Y} \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent normal RV)
then $\mathrm{a}X + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$

Agenda

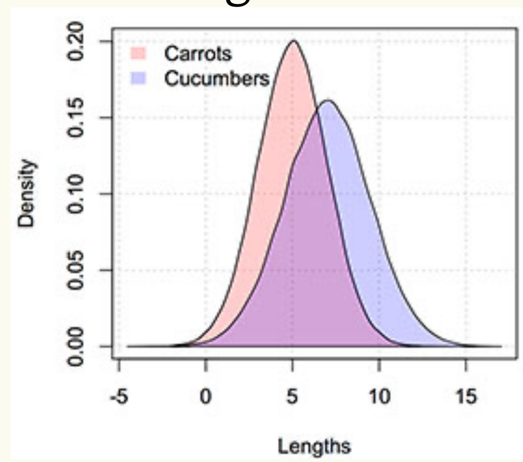
- Central Limit Theorem (CLT)
- Polling

Normal Distributions EVERYWHERE – why?

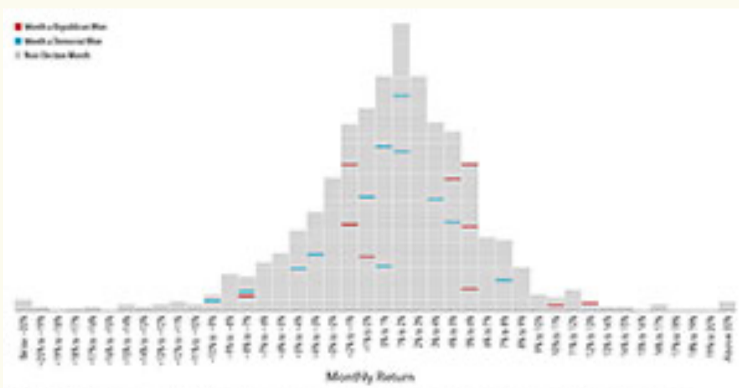
Neuron Activity



Vegetables



S&P 500 Returns after Elections



Examples from:
https://galtonboard.com/probabilityexamplesinlife

Sums of i.i.d. RVs look normal!

i.i.d. = independent and identically distributed

$X_1, \dots, X_n$ i.i.d. with expectation μ and variance σ^2

Consider $$S_n = X_1 + \cdots + X_n$$

Empirical observation:

S_n looks like a normal RV as n grows.

Central Limit Theorem

$\mathbb{E}[S_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n\mu$

$\text{Var}(S_n) = \text{Var}(X_1) + \cdots + \text{Var}(X_n) = n\sigma^2$

$X_1, \ldots, X_n$ i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$,

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

Then distribution of $Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$ converges to that of a normal distribution with mean 0 and variance 1 as $n \to \infty$.

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Theorem. (Central Limit Theorem) The CDF of Y_n converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

$$\lim_{n\to\infty} P(Y_n \le y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \mathrm{d}x$$

Summary Central Limit Theorem

$E(aX) = aE(X)$

$Var(aX) = a^2 Var(X)$

$X_1, \ldots, X_n$ i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. and $Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$

sample mean

$E(\bar{X}) = \frac{1}{n} E(\sum X_i)$, where $E(\sum X_i) = n\mu$

mean $\quad n\mu \quad\quad \mu \quad\quad 0$

$Var(\bar{X}) = Var(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n^2} Var(\sum_{i=1}^{n} X_i) = \frac{\sigma^2}{n}$, where $Var(\sum_{i=1}^{n} X_i) = n\sigma^2$

variance $\quad n\sigma^2 \quad\quad \frac{\sigma^2}{n} \quad\quad 1$

CLT: $\quad \approx N(n\mu, n\sigma^2) \quad\quad \rightarrow N(\mu, \frac{\sigma^2}{n}) \quad\quad \rightarrow N(0,1)$

CLT application

- You buy lightbulbs that burn out according to an exponential distribution with parameter $\lambda = 1.8$ lightbulbs per year.
- You buy a pack of 10 (independent) light bulbs. What is the probability that your 10-pack lasts at least 5 years?

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X_i – time for i^{th} bulb to burn out

X_i's are indep $\quad X_i \sim \exp(1.8) \Rightarrow E(X_i) = \frac{10}{1.8} \quad Var(X_i) = \frac{10}{1.8^2}$

$$P(X > 5) = P\left(\frac{X - \frac{10}{1.8}}{\sqrt{\frac{10}{1.8^2}}} > \frac{5 - \frac{10}{1.8}}{\sqrt{\frac{10}{1.8^2}}}\right) \quad \frac{5 - \frac{10}{1.8}}{\sqrt{\frac{10}{1.8^2}}} = -0.32$$

$$\underset{\text{by CLT}}{\approx} P(Z > -0.32)$$

$$= 1 - \Phi(-0.32) = \Phi(0.32)$$

$$\approx 0.62552$$

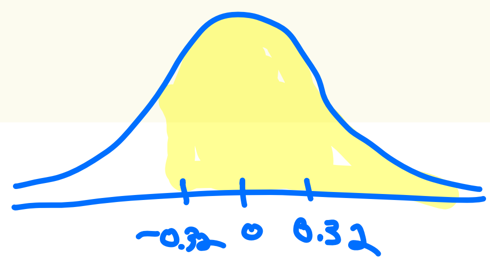
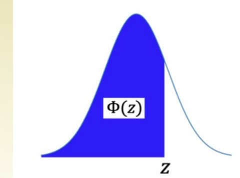


Table of Standard Cumulative Normal Density



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3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Outline of CLT steps

- Write the event you are interested in, in terms of a sum of i.i.d. random variables.
- Normalize RV to have mean 0 and standard deviation 1.
- Write event in terms of Φ, the CDF of a $\mathcal{N}(0,1)$.
- Look up in table.

Outline of CLT steps – extra step if random variables are discrete.

- Write the event you are interested in, in terms of a sum of i.i.d. random variables.
- Apply continuity correction if RVs are discrete (see yesterday's section and Tsun Section 5.7.4)
- Normalize RV to have mean 0 and standard deviation 1.
- Write event in terms of Φ, the CDF of a $\mathcal{N}(0,1)$.
- Look up in table.

Agenda

- Central Limit Theorem (CLT)
- Polling

Magic Mushrooms

Suppose there will soon be a vote on whether to legalize the therapeutic use of "magic mushrooms" and we want to conduct a poll to try to predict what will happen.

Poll to determine the fraction p of the population expected to vote in favor.

- Call up a random sample of n people to ask their opinion
- Report the empirical fraction

Questions

- Is this a good estimate?
- How to choose n?

Polling Accuracy

Often see claims that say

"Our poll found 80% support. This poll is accurate to within 5% with 98% probability"*

Will unpack what this means and how they sample enough people to know this is true.

* When it is 95% this is sometimes written as "19 times out of 20"

Formalizing Polls

Population size N, true fraction of voting in favor p, sample size n.

Problem: We don't know p, want to estimate it

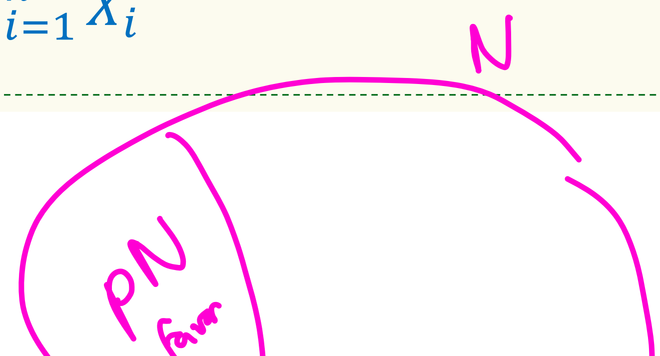
Polling Procedure

for $i = 1, \ldots, n$:

1. Pick uniformly random person to call (prob: $1/N$)
2. Ask them how they will vote

$$X_i = \begin{cases} 1, & \text{voting in favor} \\ 0, & \text{otherwise} \end{cases}$$

Report our estimate of p: $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$



Formalizing Polls

Population size N, true fraction of voting in favor p, sample size n.

Problem: We don't know p

What type of r.v. is X_i?

Poll:

	Type	$\mathbb{E}[X_i]$	$\mathrm{Var}(X_i)$
a.	Bernoulli	p	$p(1-p)$
b.	Bernoulli	p	p^2
c.	Geometric	p	$\frac{1-p}{p^2}$
d.	Binomial	np	$np(1-p)$

Polling Procedure

for $i = 1, \dots, n$:

1. Pick uniformly random person to call (prob: $1/N$)
2. Ask them how they will vote

$$X_i = \begin{cases} 1, & \text{voting in favor} \\ 0, & \text{otherwise} \end{cases}$$

Report our estimate of p: $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

Random Variables

What type of r.v. is X_i?

Type	$\mathbb{E}[X_i]$	$\mathrm{Var}(X_i)$
Bernoulli	p	$p(1-p)$

What about $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$?

Poll:

	$\mathbb{E}[\bar{X}]$	$\mathrm{Var}(\bar{X})$
a.	np	$np(1-p)$
b.	p	$p(1-p)$
c.	p	$p(1-p)/n$
d.	p/n	$p(1-p)/n$

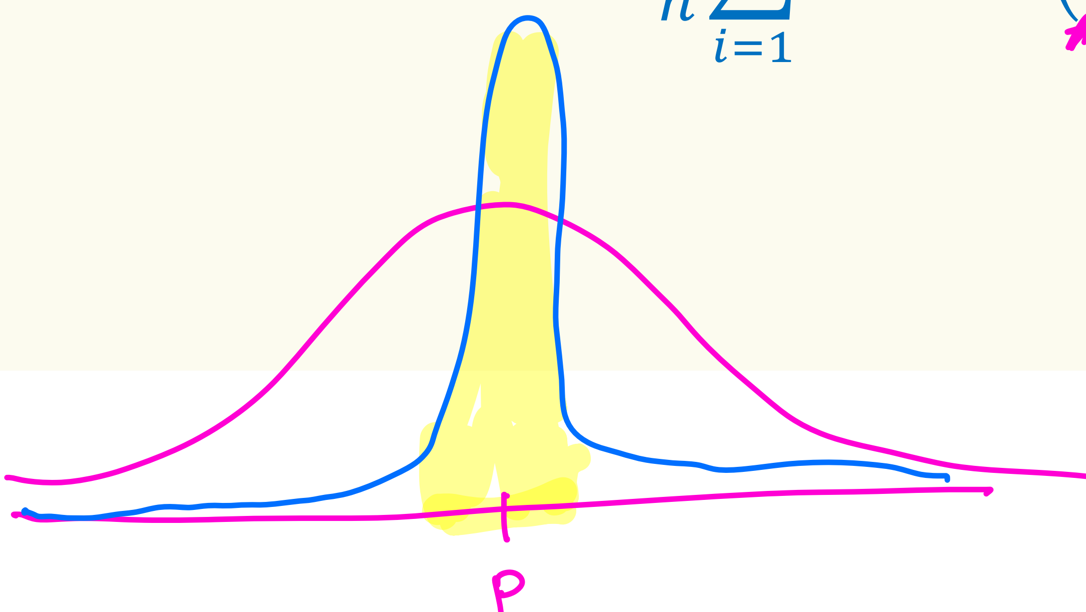
Central Limit Theorem

With i.i.d random variables $X_1, X_2, \ldots, X_n$ where $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$

variance

As $n \to \infty$,

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathcal{N}\left(p, \frac{p(1-p)}{n}\right)$$

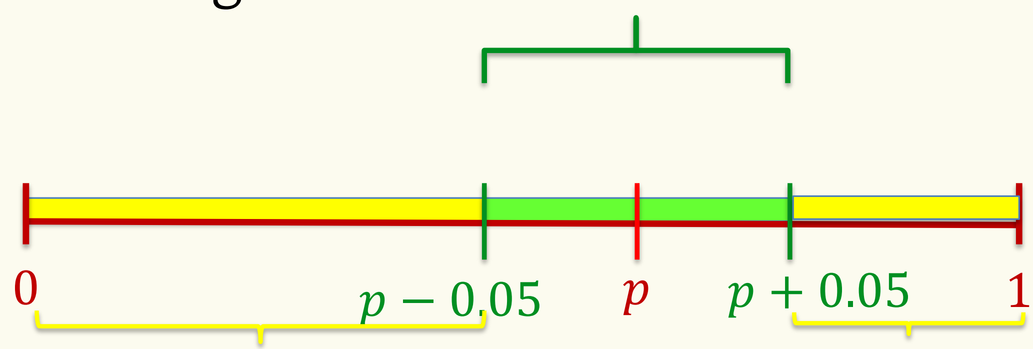


Roadmap: Bounding Error

with prob ≥ 0.98

Goal: Find the value of n such that 98% of the time, the estimate $\overline{X}$ is within 5% of the true p

Get good estimate if $\overline{X}$ lands in this region

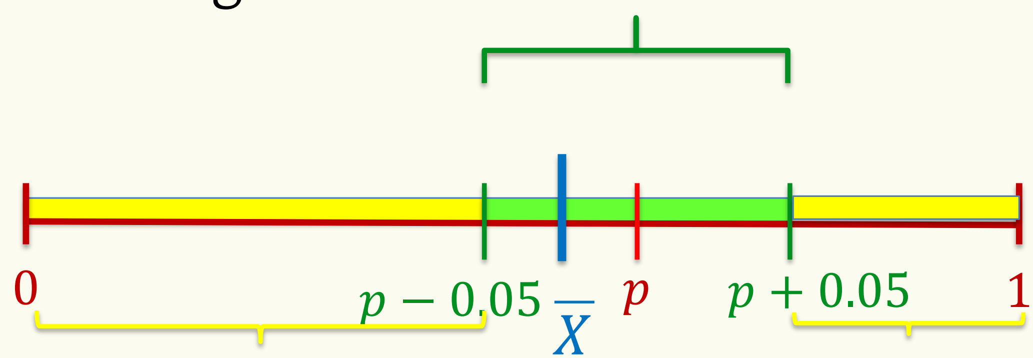


How does the value of n play a role?

Roadmap: Bounding Error

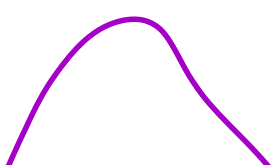
Goal: Find the value of n such that 98% of the time, the estimate $\bar{X}$ is within 5% of the true p

Get good estimate if $\bar{X}$ lands in this region

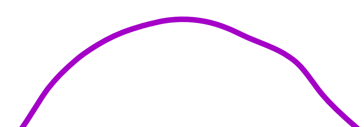


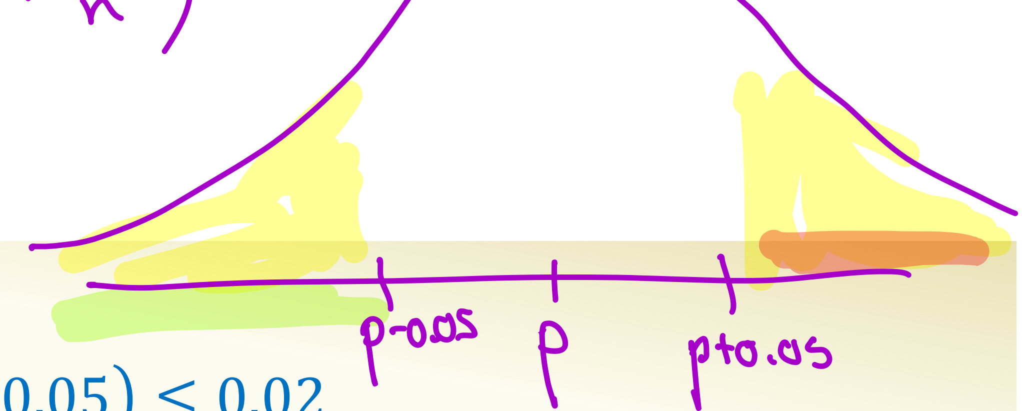
Question: for what n is $P(|\bar{X} - p| > 0.05) \leq 0.02$

prob $\bar{X}$ lands in yellow



$\bar{X} \sim N\left(p, \frac{p(1-p)}{\ldots}\right)$





Question: for what n is $P(|\overline{X} - p| > 0.05) \leq 0.02$

prob $\overline{X}$ in yellow

$$Pr(\overline{X} - p > 0.05) + P(\overline{X} - p < -0.05)$$

$$= 2P(\overline{X} - p > 0.05)$$

$$P(\overline{X} - p > 0.05) = P\left(\frac{\overline{X} - p}{\sqrt{\frac{p(1-p)}{n}}} > \frac{0.05}{\sqrt{\frac{p(1-p)}{n}}}\right)$$

$\frac{\overline{X} - p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$, $\quad \frac{0.05}{\sqrt{\frac{p(1-p)}{n}}} = a$

$$2P(Z > a) \leq 0.02$$

$$P(Z > a) \leq 0.01$$

true if $a \geq 2.33$

Recall!!

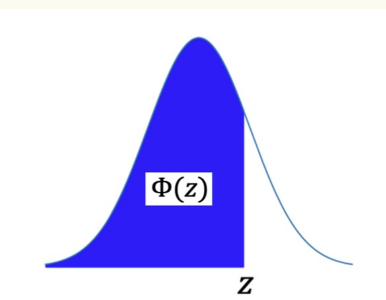
For what a is

$$P(Z > a) \leq 0.01 \ ?$$

Equivalently

$$P(Z \leq a) \geq 0.99 \ ?$$

For any $a \geq 2.33$
$P(Z > a) \leq 0.01$.



Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
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0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
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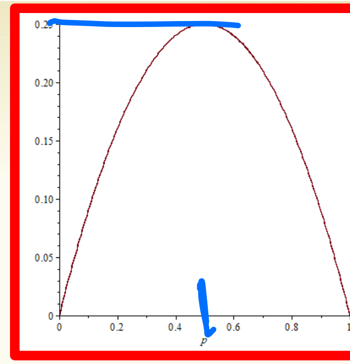
$$\frac{0.05}{\sqrt{\frac{p(1-p)}{n}}} \geq 2.33$$

graph of $p(1-p)$

$\frac{1}{4}$



$p = \frac{1}{2}$

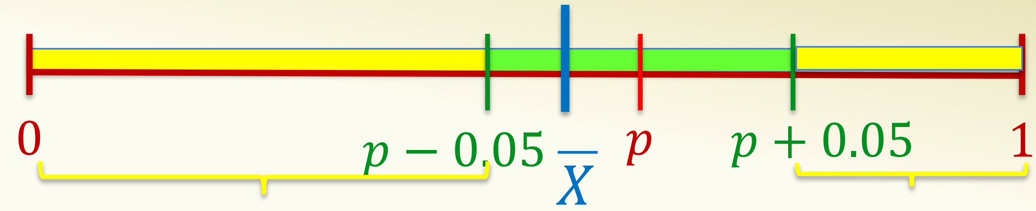
$$\equiv \quad \boxed{\sqrt{n} \geq \frac{2.33\sqrt{p(1-p)}}{0.05}}$$

as long as

$$\sqrt{n} \geq \frac{2.33\sqrt{\frac{1}{4}}}{0.05}$$

satisfy $\forall p \in [0,1]$

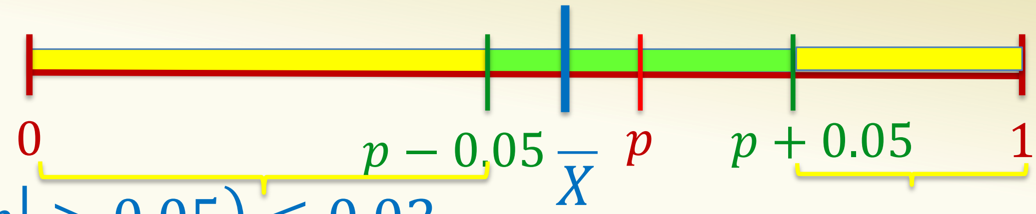
Recap



Goal: Find the value of n such that 98% of the time, the estimate $\bar{X}$ is within 5% of the true p

1. Define question. For what n is $P(|\bar{X} - p| > 0.05) \leq 0.02$
2. Apply CLT: By CLT $\bar{X} \to \mathcal{N}(\mu, \sigma^2)$ where $\mu = p$ and $\sigma^2 = p(1-p)/n$
3. Convert to a standard normal. Specifically, define $Z = \frac{\bar{X}-\mu}{\sigma} = \frac{\bar{X}-p}{\sigma}$. Then, by the CLT $Z \to \mathcal{N}(0,1)$
4. Solve for n

In more detail



1. The question: for what n is $P\left(\left|\overline{X}-p\right|>0.05\right)\leq 0.02$

2. By CLT $\overline{X}\rightarrow\mathcal{N}(\mu,\sigma^2)$ where $\mu=p$ and $\sigma^2=p(1-p)/n$

3. $P\left(\left|\overline{X}-p\right|>0.05\right)=P\left(\overline{X}-p>0.05\right)+P\left(\overline{X}-p<-0.05\right)$

4. $P\left(\left|\overline{X}-p\right|>0.05\right)\leq 0.02$ equivalent to $P\left(\overline{X}-p>0.05\right)\leq 0.01$

5. Equivalent to $P\left({}^{\overline{X}-p}\Big/_{\frac{\sqrt{p(1-p)}}{\sqrt{n}}}>{}^{0.05}\Big/_{\frac{\sqrt{p(1-p)}}{\sqrt{n}}}\right)\leq 0.01$

6. Equivalent to $P\left(Z>\frac{0.05}{\sqrt{p(1-p)}/\sqrt{n}}\right)\leq 0.01$

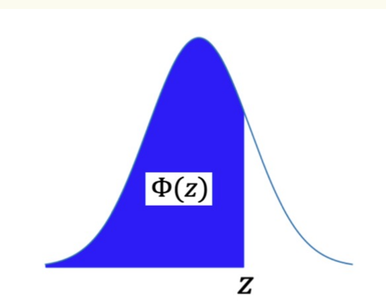
Recall!!

For what a is

$P(Z > a) \leq 0.01$?

For any $a \geq 2.33$

$P(Z > a) \leq 0.01.$



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1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
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2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
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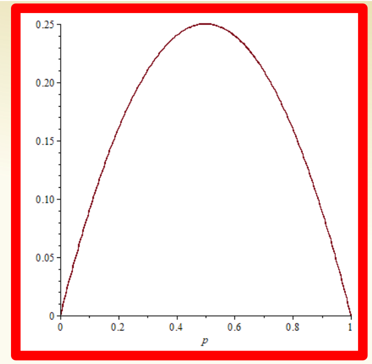

So we know that if $\frac{0.05}{\sqrt{p(1-p)}/\sqrt{n}} \geq 2.33$

Then $P\left(Z > \frac{0.05}{\sqrt{p(1-p)}/\sqrt{n}}\right) \leq 0.01$.

So just need to solve this: $\sqrt{n} \geq \frac{2.33\sqrt{p(1-p)}}{0.05}$

Since we don't know p, we will just take n so it works for all p. RHS is largest with p = 0.5. So we take

$\sqrt{n} \geq \frac{2.33 \cdot 0.5}{0.05}$ or $\sqrt{n} \geq 23.3$

Then $n \geq 543 \geq (23.3)^2$ would be good enough.

- Since only have $Z \to \mathcal{N}(0,1)$ so there is some loss due to approximation error.
- So should increase n a bit more to be safe.

Zooming back out: we found an approximate``confidence interval"

We are trying to estimate some parameter (e.g. p). We output an estimator $\overline{X}$ such that $P\left(\left|\overline{X}-p\right|>\epsilon\right)\leq\delta$ for some (ϵ,δ).

- Often found using CLT
- We say that we are $(1-\delta)$*100% confident that the result of our poll ($\overline{X}$) is an accurate estimate of p to within ϵ*100% percent.
- In our example, $(\epsilon=0.05,\delta=0.02)$.

Idealized Polling

So far, we have been discussing "idealized polling". Real life is normally not so nice ☹

Assumed we can sample people uniformly at random, not really possible in practice

- Not everyone responds
- Response rates might differ in different groups
- Will people respond truthfully?

Makes polling in real life much more complex than this idealized model!