CSE 312

Foundations of Computing II

21: Maximum Likelihood Estimation (MLE)

Agenda

- Wrap up on Law of Total Expectation and Law of Total Probability
- Idea: Estimation
- Maximum Likelihood Estimation (example: mystery coin)
- Continuous MLE

Conditional Expectation

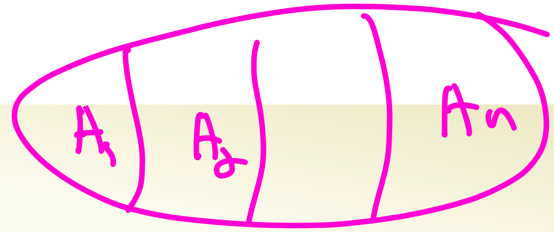
Definition. If X is a discrete random variable then the **conditional expectation** of X given event A is

$$\mathbb{E}[X \mid A] = \sum_{x \in \Omega_X} x \cdot P(X = x \mid A)$$

Note:

- Linearity of expectation still applies here

$$\mathbb{E}[aX + bY + c \mid A] = a\, \mathbb{E}[X \mid A] + b\, \mathbb{E}[Y \mid A] + c$$



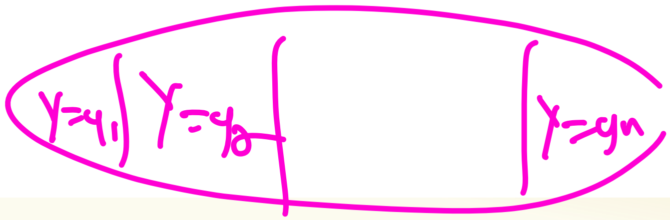
Law of Total Expectation

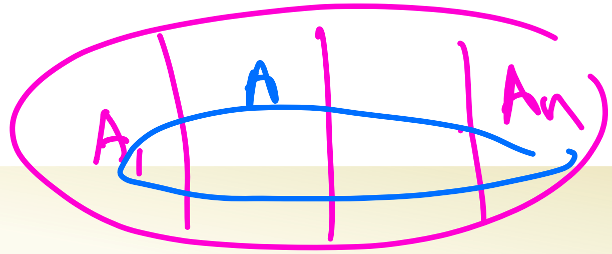
Law of Total Expectation (event version). Let X be a random variable and let events $A_1, \dots, A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \cdot P(A_i)$$

Law of Total Expectation (random variable version). Let X be a random variable and Y be a discrete random variable. Then,

$$\mathbb{E}[X] = \sum_{y \in \Omega_Y} \mathbb{E}[X \mid Y = y] \cdot P(Y = y)$$





Law of total probability

Definition. Let A be an event and Y a discrete random variable. Then

$$P[A] = \sum_{y \in \Omega_Y} P(A|Y = y) p_Y(y)$$

Definition. Let A be an event and Y a continuous random variable. Then

$$P[A] = \int_{-\infty}^{\infty} P(A|Y = y) f_Y(y) \mathrm{d}y$$

Example use of law of total probability

Suppose that the time until server 1 crashes is $X \sim Exp\ (\lambda)$ and the time until server 2 crashes is independent, with $Y \sim Exp\ (\mu)$.

What is the probability that server 1 crashes before server 2?

$P(X < Y)$

$P(Y > x) = 1 - P(Y \le x)$

$P(Y>X) = \sum_{x \in \mathcal{X}} P(Y>X \mid X=x) \frac{Pr(X=x)}{P_X(x)}$

Example use of law of total probability

$X \sim Exp\ (\lambda), Y \sim Exp\ (\mu)$.

What is the probability that $Y > X$?

$\forall \lambda \quad \int_0^\infty \lambda e^{-\lambda x} dx = 1$

$$P(Y > X) = \int_0^\infty \Pr(Y > X \mid X = x)\ f_X(x) dx$$

$$= \int_0^\infty \Pr(Y > x \mid X = x)\ \lambda e^{-\lambda x}\ dx$$

indep of X & Y

$$= \int_0^\infty \Pr(Y > x) \lambda e^{-\lambda x}\ dx$$

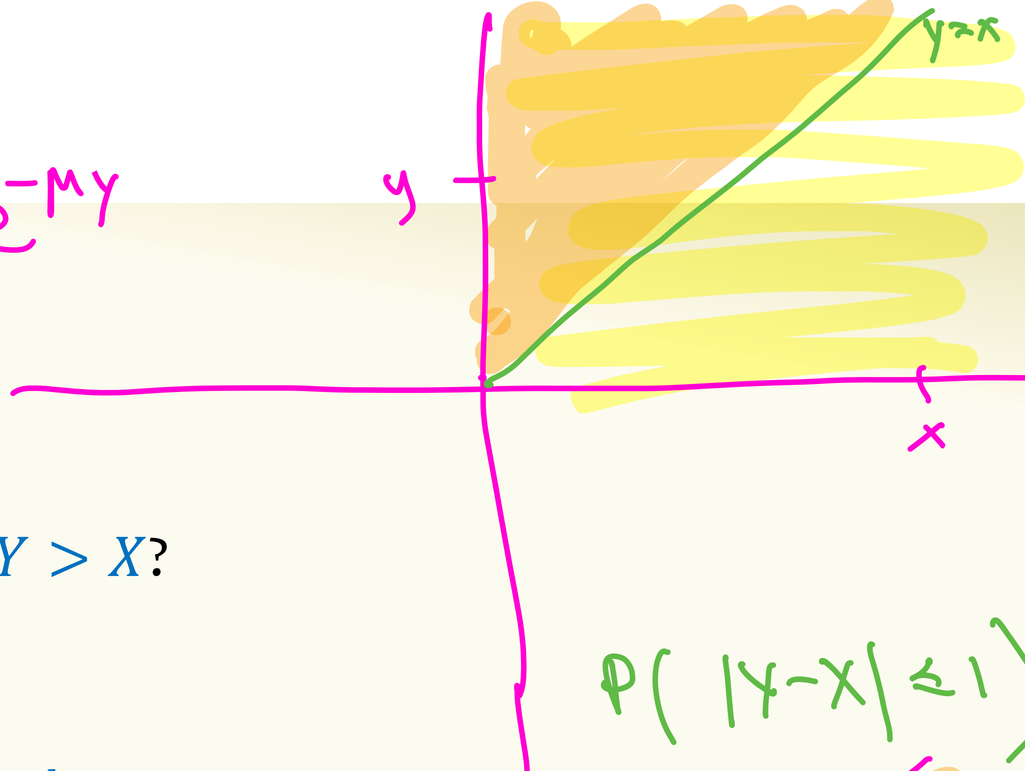
$$= \int_0^\infty e^{-\mu x}\ \lambda\ e^{-\lambda x}\ dx$$

$$= \lambda \int_0^\infty e^{-(\mu+\lambda)x} dx$$

$$= \frac{\lambda}{\lambda + \mu} \int_0^\infty (\lambda + \mu) \cdot e^{-\mu x}\ e^{-\lambda x}\ dx$$

$$= \frac{\lambda}{\lambda + \mu}$$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$
$$= \lambda \mu e^{-\lambda x} e^{-\mu y}$$



Alternative approach

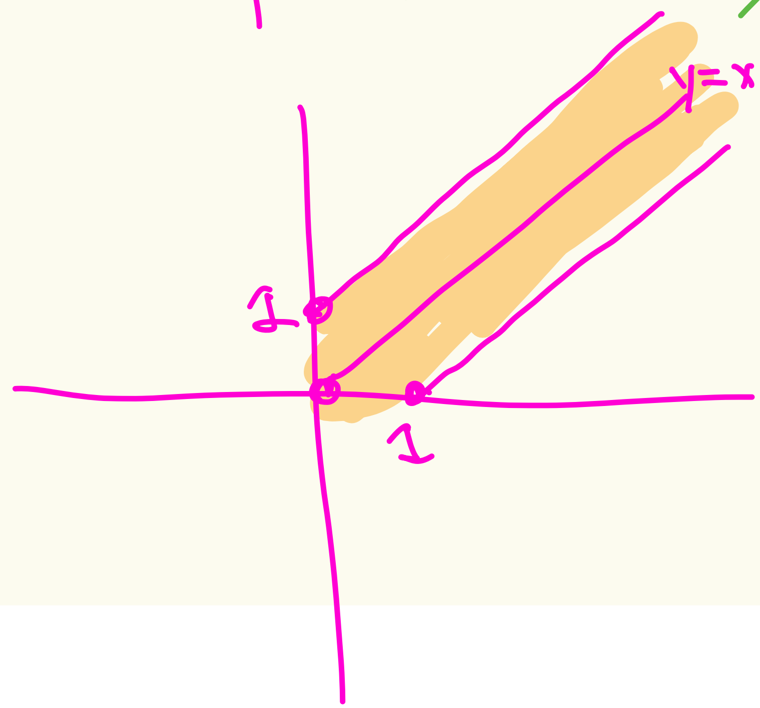
$X \sim Exp\ (\lambda), Y \sim Exp\ (\mu)$.

What is the probability that $Y > X$?

$$P(Y > X) = \int_{x=0}^{\infty} \int_{y=x}^{\infty} f_{X,Y}(x,y) \mathrm{d}y\, \mathrm{d}x$$

$$= \int_{x=0}^{\infty} \int_{y=x}^{\infty} f_X(x) \cdot f_Y(y) \mathrm{d}y\, \mathrm{d}x$$

$P(|Y-X| \leq 1)$



Reference Sheet (with continuous RVs)

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = P(X = x, Y = y)$	$f_{X,Y}(x,y) \neq P(X = x, Y = y)$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \leq x} \sum_{s \leq y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s) ds dt$
Normalization	$\sum_{x} \sum_{y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_{y} p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$E[g(X,Y)] = \sum_{x} \sum_{y} g(x,y) p_{X,Y}(x,y)$	$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Conditional PMF/PDF	$p_{X \mid Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$E[X \mid Y = y] = \sum_{x} x p_{X \mid Y}(x \mid y)$	$E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X \mid Y}(x \mid y) dx$
Independence	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$

Agenda

- Idea: Estimation
- Maximum Likelihood Estimation (example: mystery coin)
- Continuous MLE

Probability vs Statistics

$\text{Ber}(p = 0.5)$ ⇒ **Probability** Given model, predict data ⇒ $P(THHTHH)$

$\text{Ber}(p = ??)$ ⇐ **Statistics** Given data, predict model ⇐ $THHTHH$

Recap Formalizing Polls

We assume that poll answers $X_1, \dots, X_n \sim \text{Ber}(p)$ i.i.d. for unknown p

Goal: Estimate p

We did this by computing $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i$

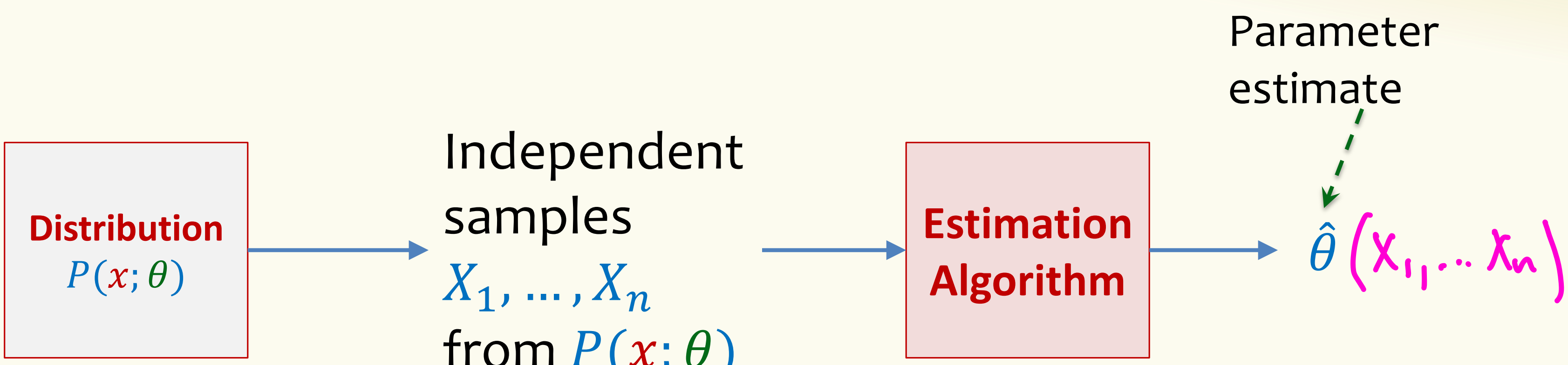
Recap More generally …

In estimation we often … .

- Assume: we know the type of the random variable that we are observing independent samples from
  - We just don't know the parameters, e.g.
    - the bias p of a random coin $\text{Bernoulli}(p)$
    - The arrival rate λ for the $\text{Poisson}(\lambda)$ or $\text{Exponential}(\lambda)$
    - The mean μ and variance σ of a normal $\mathcal{N}(\mu, \sigma)$

- Goal: find the "best" parameters to fit the data

Statistics: Parameter Estimation – Workflow



θ = unknown parameter

Example: coin flip distribution with unknown θ = probability of heads

Observation: $HTTHHHTHTHTTTTHTHTTTTTHT$

Goal: Estimate θ

Example

Suppose we have a mystery coin with some probability p of coming up heads. We flip the coin 8 times, independent of other flips, and see the following sequence of flips

$$TTHTHTTH$$

Given this data, what would you estimate p is?

$\frac{3}{8}$

How can you argue "objectively" that this your estimate is the best estimate?

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Likelihood

Say we see outcome $HHTHH$.

You tell me your best guess about the value of the unknown parameter θ (a.k.a. p) is 4/5. Is there some way that you can argue "objectively" that this is the best estimate?

What is "likelihood" of seeing HHTHH if unknown param is θ?

$\mathcal{L} \ldots = \theta^4(1-\theta) = \theta^4 - \theta^5$

What θ maximizes this fn?

$$\frac{d}{d\theta}\mathcal{L}(\theta) = 4\theta^3 - 5\theta^4 = 0$$

$$4\theta^3 = 5\theta^4$$

$$\frac{4}{5} = \theta$$

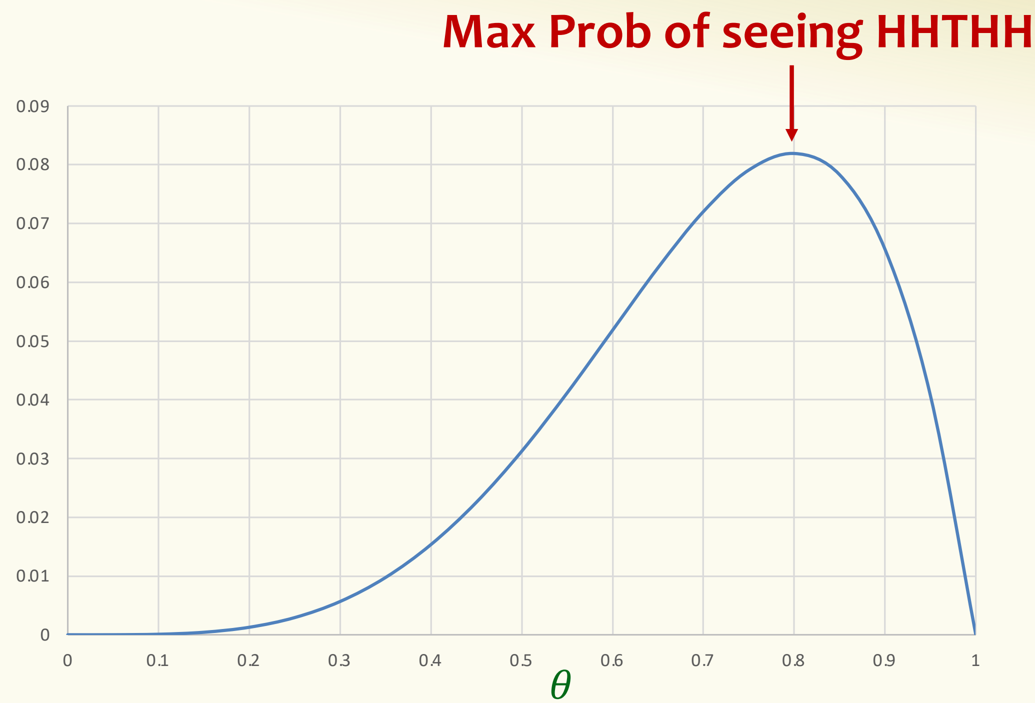
Likelihood

Say we see outcome $HHTHH$.

$$\mathcal{L}(HHTHH\ ;\ \theta) = \theta^4(1-\theta)$$

Probability of observing the outcome $HHTHH$ if $\theta =$ prob. of heads.

For a fixed outcome $HHTHH$, this is a function of θ.



Likelihood of Different Observations

(Discrete case)

Definition. The **likelihood** of independent observations $x_1, \ldots, x_n$ is

$$\mathcal{L}(x_1, x_2, \ldots, x_n\,; \theta) = \prod_{i=1}^{n} P(x_i; \theta)$$

Example:
Say we see outcome $HHTHH$.

$$\mathcal{L}(HHTHH\,; \theta) = P(H; \theta) \cdot P(H; \theta) \cdot P(T; \theta) \cdot P(H; \theta) \cdot P(H; \theta) = \theta^4(1-\theta)$$

Likelihood vs. Probability

- Fixed θ: **probability** $\prod_{i=1}^{n} P(x_i; \theta)$ that dataset $x_1, \ldots, x_n$ is sampled by distribution with parameter θ
  - A function of $x_1, \ldots, x_n$
- Fixed $x_1, \ldots, x_n$: **likelihood** $\mathcal{L}(x_1, x_2, \ldots, x_n\ ; \theta)$ that parameter θ explains dataset $x_1, \ldots, x_n$.
  - A function of θ

These notions are the same number if we fix *both* $x_1, \ldots, x_n$ and θ, but different role/interpretation

Likelihood of Different Observations

(Discrete case)

Definition. The **likelihood** of independent observations $x_1, \dots, x_n$ is

$$\mathcal{L}(x_1, x_2, \dots, x_n\, ; \theta) = \prod_{i=1}^{n} P(x_i; \theta)$$

Maximum Likelihood Estimation (MLE). Given data $x_1, \dots, x_n$, find $\hat{\theta}$ such that $\mathcal{L}\left(x_1, x_2, \dots, x_n\, ; \hat{\theta}\right)$ is maximized!

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\ \mathcal{L}(x_1, x_2, \dots, x_n\, ; \theta)$$

Example – Coin Flips

H H T H...

Observe: Coin-flip outcomes $x_1, \dots, x_n$, with n_H heads, n_T tails

– i.e., $n_H + n_T = n$

Goal: estimate θ = prob. heads.

$$\mathcal{L}(x_1, \dots, x_n ; \theta) = \theta^{n_H} (1-\theta)^{n_T}$$

Goal: find θ that maximizes $\mathcal{L}(x_1, \dots, x_n ; \theta)$

Example – Coin Flips

Observe: Coin-flip outcomes $x_1, \ldots, x_n$, with n_H heads, n_T tails

– i.e., $n_H + n_T = n$

Goal: estimate θ = prob. heads.

$$\mathcal{L}(x_1, \ldots, x_n ; \theta) = \theta^{n_H} (1 - \theta)^{n_T}$$

$$\frac{\partial}{\partial \theta} \mathcal{L}(x_1, \ldots, x_n ; \theta) = ???$$

While it is possible to compute this derivative, it's not always nice since we are working with products.

ln ℒ is maximized at same θ ℒ is

Log-Likelihood

We can save some work if we use the **log-likelihood** instead of the likelihood directly.

Definition. The **log-likelihood** of independent observations $x_1, \dots, x_n$ is

$$\ln \mathcal{L}(x_1, \dots, x_n\,; \theta) = \ln \prod_{i=1}^{n} P(x_i; \theta) = \sum_{i=1}^{n} \ln P(x_i; \theta)$$

Useful log properties

$$\ln(ab) = \ln(a) + \ln(b)$$
$$\ln(a/b) = \ln(a) - \ln(b)$$
$$\ln\left(a^b\right) = b \cdot \ln(a)$$

Example – Coin Flips

$\ln(ab) = \ln(a) + \ln(b)$
$\ln(a/b) = \ln(a) - \ln(b)$
$\ln(a^b) = b \cdot \ln(a)$

Observe: Coin-flip outcomes $x_1, \dots, x_n$, with n_H heads, n_T tails

– i.e., $n_H + n_T = n$

Goal: estimate θ = prob. heads.

$$\mathcal{L}(x_1, \dots, x_n ; \theta) = \theta^{n_H} (1 - \theta)^{n_T}$$

$$\ln \mathcal{L} = \ln \theta^{n_H} + \ln (1-\theta)^{n_T}$$

$$= n_H \ln \theta + n_T \ln(1-\theta)$$

Example – Coin Flips

$\frac{d}{dx} \ln x = \frac{1}{x}$

Observe: Coin-flip outcomes $x_1, \dots, x_n$, with n_H heads, n_T tails
– i.e., $n_H + n_T = n$

Goal: estimate θ = prob. heads.

$$\mathcal{L}(x_1, \dots, x_n ; \theta) = \theta^{n_H} (1-\theta)^{n_T}$$

$$\ln \mathcal{L}(x_1, \dots, x_n ; \theta) = n_H \ln \theta + n_T \ln(1-\theta)$$

$$\frac{\partial}{\partial \theta} \ln \mathcal{L}(x_1, \dots, x_n ; \theta) = n_H \cdot \frac{1}{\theta} - n_T \cdot \frac{1}{1-\theta}$$

Want value $\hat{\theta}$ of θ s.t. $\frac{\partial}{\partial \theta} \ln \mathcal{L}(x_1, \dots, x_n ; \theta) = 0$

So we need $n_H \cdot \frac{1}{\hat{\theta}} - n_T \cdot \frac{1}{1-\hat{\theta}} = 0$

Solving gives
$$\hat{\theta} = \frac{n_H}{n}$$

General Recipe

1. **Input** Given n i.i.d. samples $x_1, \dots, x_n$ from parametric model with parameter θ.
2. **Likelihood** Define your likelihood $\mathcal{L}(x_1, \dots, x_n ; \theta)$.
   – For discrete $\mathcal{L}(x_1, \dots, x_n ; \theta) = \prod_{i=1}^{n} P(x_i ; \theta)$
3. **Log** Compute $\ln \mathcal{L}(x_1, \dots, x_n ; \theta)$
4. **Differentiate** Compute $\frac{\partial}{\partial \theta} \ln \mathcal{L}(x_1, \dots, x_n ; \theta)$
5. **Solve for** $\hat{\theta}$ by setting derivative to 0 and solving for max.

Generally, you need to do a second derivative test to verify it is a maximum, but we won't ask you to do that in CSE 312.

Brain Break

Agenda

- Idea: Estimation
- Maximum Likelihood Estimation (example: mystery coin)
- Continuous MLE

The Continuous Case

Given n (independent) samples $x_1, \dots, x_n$ from (continuous) parametric model $f(x_i; \theta)$ which is now a family of densities

Definition. The **likelihood** of independent observations $x_1, \dots., x_n$ is

$$\mathcal{L}(x_1, \dots, x_n\,; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$$

Replace pmf with pdf!

$Pr(X \approx x) = f_X(x)\,dx$

Why density?

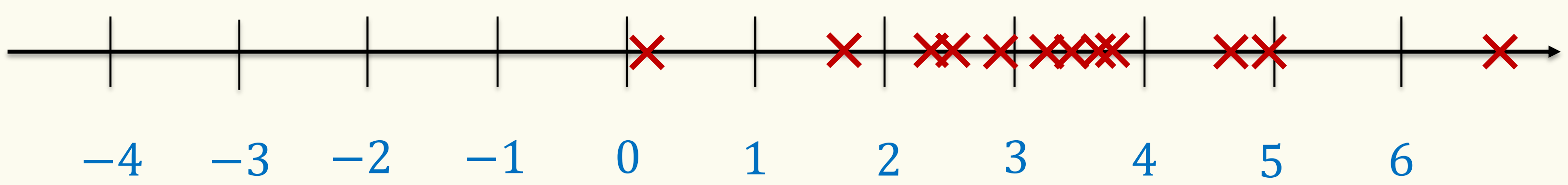
- Density ≠ probability, but:
  - For maximizing likelihood, we really only care about relative likelihoods, and density captures that
  - has desired property that likelihood increases with better fit to the model

Agenda

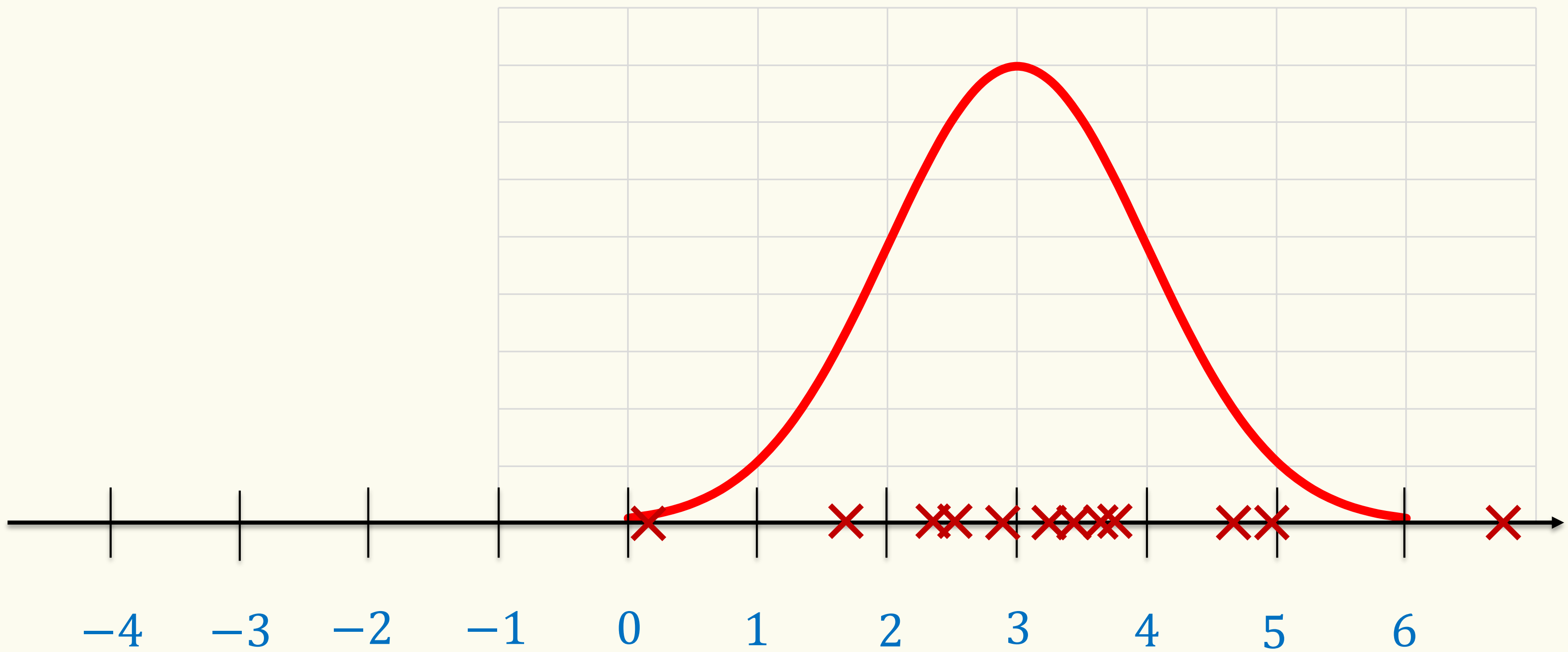
- MLE for Normal Distribution
- Unbiased and Consistent Estimators
- Odds and ends

n samples $x_1, \dots, x_n \in \mathbb{R}$ from Gaussian $\mathcal{N}(\mu, 1)$. Most likely μ?

[i.e., we are given the promise that the variance is 1]



n samples $x_1, \ldots, x_n \in \mathbb{R}$ from Gaussian $\mathcal{N}(\mu, 1)$. Most likely μ?



Example – Gaussian Parameters

$\ln(ab) = \ln(a) + \ln(b)$
$\ln(a/b) = \ln(a) - \ln(b)$
$\ln(a^b) = b \cdot \ln(a)$

Normal outcomes $x_1, \dots, x_n$, known variance $\sigma^2 = 1$

Goal: estimate θ, the expectation

$$\mathcal{L}(x_1, \dots, x_n ; \theta) = \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i-\theta)^2}{2}} \right) = \left(\frac{1}{\sqrt{2\pi}} \right)^n \prod_{i=1}^{n} e^{-\frac{(x_i-\theta)^2}{2}}$$

$$\ln \mathcal{L}(x_1, \dots, x_n ; \theta) = -n \frac{\ln 2\pi}{2} - \sum_{i=1}^{n} \frac{(x_i - \theta)^2}{2}$$

Example – Gaussian Parameters

Goal: estimate θ = expectation

Normal outcomes $x_1, \dots, x_n$, known variance $\sigma^2 = 1$

$$\ln \mathcal{L}(x_1, \dots, x_n ; \theta) = -n \frac{\ln 2\pi}{2} - \sum_{i=1}^{n} \frac{(x_i - \theta)^2}{2}$$

Note: $\frac{\partial}{\partial \theta} \frac{(x_i - \theta)^2}{2} = \frac{1}{2} \cdot 2 \cdot (x_i - \theta) \cdot (-1) = \theta - x_i$

Example – Gaussian Parameters

Goal: estimate θ = expectation

Normal outcomes $x_1, \ldots, x_n$, known variance $\sigma^2 = 1$

$$\ln \mathcal{L}(x_1, \ldots, x_n ; \theta) = -n\frac{\ln 2\pi}{2} - \sum_{i=1}^{n} \frac{(x_i - \theta)^2}{2}$$

Note: $\frac{\partial}{\partial \theta}\frac{(x_i-\theta)^2}{2} = \frac{1}{2} \cdot 2 \cdot (x_i - \theta) \cdot (-1) = \theta - x_i$

$$\frac{\partial}{\partial \theta} \ln \mathcal{L}(x_1, \ldots, x_n ; \theta) = \sum_{i=1}^{n} (x_i - \theta) = \left(\sum_{i=1}^{n} x_i\right) - n\theta$$

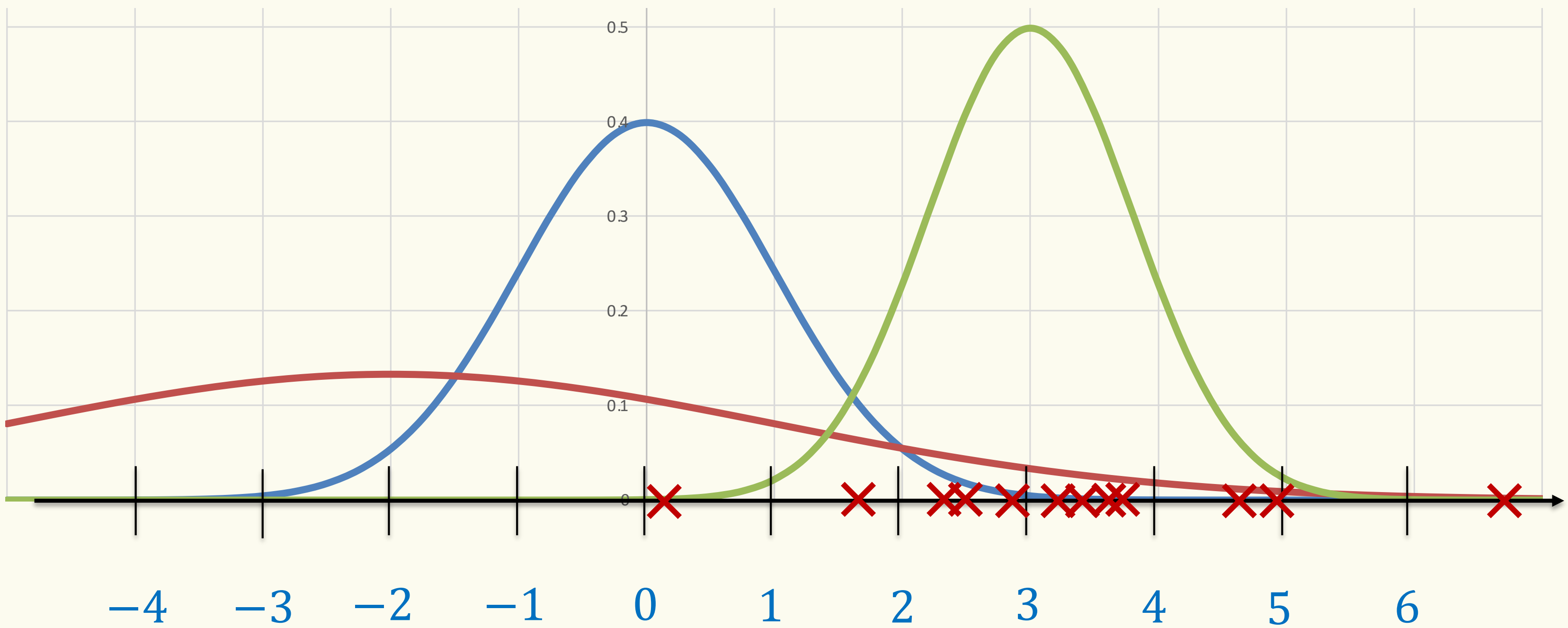
$x_1 - \theta + x_2 - \theta + \ldots + x_n - \theta$

So… solve $\sum_{i=1}^{n} x_i - n\hat{\theta} = 0$ for $\hat{\theta}$

$$\hat{\theta} = \frac{\sum_i^n x_i}{n}$$

In other words, MLE is the *sample mean* of the data.

Next: n samples $x_1, \dots, x_n \in \mathbb{R}$ from Gaussian $\mathcal{N}(\mu, \sigma^2)$.
Most likely μ and σ^2?

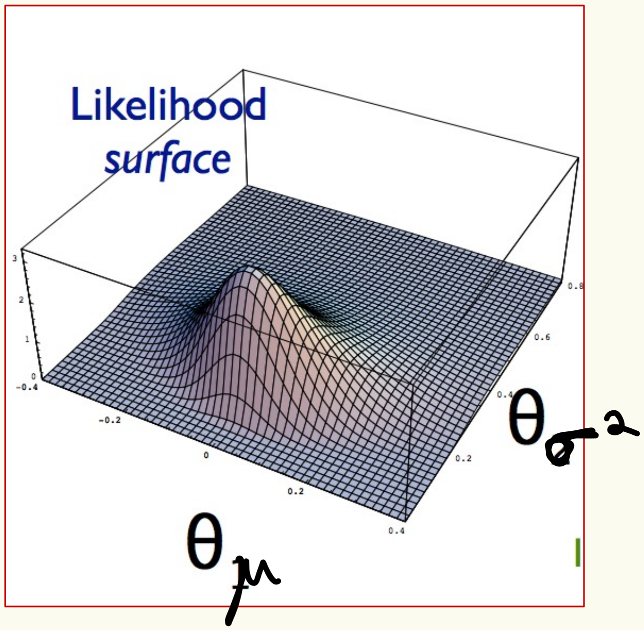


$\ln(ab) = \ln(a) + \ln(b)$
$\ln(a/b) = \ln(a) - \ln(b)$
$\ln(a^b) = b \cdot \ln(a)$

Two-parameter optimization

Normal outcomes $x_1, \dots, x_n$

Goal: estimate θ_μ = expectation and θ_{σ^2} = variance



$$\mathcal{L}(x_1, \dots, x_n\,; \theta_\mu, \theta_{\sigma^2}) = \left(\frac{1}{\sqrt{2\pi\theta_{\sigma^2}}}\right)^n \prod_{i=1}^{n} e^{-\frac{(x_i - \theta_\mu)^2}{2\theta_{\sigma^2}}}$$

$$\ln \mathcal{L}(x_1, \dots, x_n\,; \theta_\mu, \theta_{\sigma^2}) =$$

$$= -n\frac{\ln(2\pi\,\theta_{\sigma^2})}{2} - \sum_{i=1}^{n} \frac{(x_i - \theta_\mu)^2}{2\theta_{\sigma^2}}$$

Two-parameter estimation

$$\ln \mathcal{L}\left(x_1, \dots, x_n ; \theta_\mu, \theta_{\boldsymbol{\sigma}^2}\right) = -\frac{\ln(2\pi\, \theta_{\boldsymbol{\sigma}^2})}{2} - \sum_{i=1}^{n} \frac{\left(x_i - \theta_\mu\right)^2}{2\theta_{\boldsymbol{\sigma}^2}}$$

Find pair $\hat{\theta}_\mu, \hat{\theta}_{\boldsymbol{\sigma}^2}$ that maximizes $\ln \mathcal{L}\left(x_1, \dots, x_n ; \theta_\mu, \theta_{\boldsymbol{\sigma}^2}\right)$

$$\frac{\partial}{\partial \theta_\mu} \ln \mathcal{L} = 0$$

$$\frac{\partial}{\partial \theta_{\sigma^2}} \ln \mathcal{L} = 0$$

Two-parameter estimation

$$\ln \mathcal{L}(x_1, \ldots, x_n ; \theta_\mu, \theta_{\boldsymbol{\sigma}^2}) = -\frac{\ln(2\pi\, \theta_{\boldsymbol{\sigma}^2})}{2} - \sum_{i=1}^{n} \frac{(x_i - \theta_\mu)^2}{2\theta_{\boldsymbol{\sigma}^2}}$$

We need to find a solution $\hat{\theta}_\mu, \hat{\theta}_{\boldsymbol{\sigma}^2}$ to

$$\frac{\partial}{\partial \theta_\mu} \ln \mathcal{L}(x_1, \ldots, x_n ; \theta_\mu, \theta_{\boldsymbol{\sigma}^2}) = 0$$

$$\frac{\partial}{\partial \theta_{\boldsymbol{\sigma}^2}} \ln \mathcal{L}(x_1, \ldots, x_n ; \theta_\mu, \theta_{\boldsymbol{\sigma}^2}) = 0$$

MLE for Expectation

$$\ln \mathcal{L}(x_1, \dots, x_n ; \theta_\mu, \theta_{\boldsymbol{\sigma}^2}) = -n \frac{\ln(2\pi\, \theta_{\boldsymbol{\sigma}^2})}{2} - \sum_{i=1}^{n} \frac{(x_i - \theta_\mu)^2}{2\theta_{\boldsymbol{\sigma}^2}}$$

$$\frac{\partial}{\partial \theta_\mu} \ln \mathcal{L}(x_1, \dots, x_n ; \theta_\mu, \theta_{\boldsymbol{\sigma}^2}) = \frac{1}{\theta_{\boldsymbol{\sigma}^2}} \sum_{i}^{n} (x_i - \theta_\mu) = 0$$

MLE for Expectation

$$\ln \mathcal{L}(x_1, \dots, x_n ; \theta_\mu, \theta_{\sigma^2}) = -n\frac{\ln(2\pi\,\theta_{\sigma^2})}{2} - \sum_{i=1}^{n} \frac{(x_i - \theta_\mu)^2}{2\theta_{\sigma^2}}$$

$$\frac{\partial}{\partial \theta_\mu} \ln \mathcal{L}(x_1, \dots, x_n ; \theta_\mu, \theta_{\sigma^2}) = \frac{1}{\theta_{\sigma^2}} \sum_{i}^{n} (x_i - \theta_\mu) = 0$$

$$\hat{\theta}_\mu = \frac{\sum_i^n x_i}{n}$$

In other words, MLE of expectation is (again) the *sample mean* of the data, regardless of θ_{σ^2}

What about the variance?

MLE for Variance

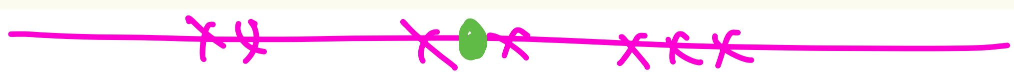
$$\ln \mathcal{L}\left(x_1, \ldots, x_n\, ; \hat{\theta}_\mu, \theta_{\sigma^2}\right) = -n \frac{\ln(2\pi\, \theta_{\sigma^2})}{2} - \sum_{i=1}^{n} \frac{\left(x_i - \hat{\theta}_\mu\right)^2}{2\theta_{\sigma^2}}$$

$$= -n \frac{\ln 2\pi}{2} - n \frac{\ln \theta_{\sigma^2}}{2} - \frac{1}{2\theta_{\sigma^2}} \sum_{i=1}^{n} \left(x_i - \hat{\theta}_\mu\right)^2$$

$$\frac{\partial}{\partial \theta_{\sigma^2}} \ln \mathcal{L}\left(x_1, \ldots, x_n\, ; \hat{\theta}_\mu, \theta_{\sigma^2}\right) = -\frac{n}{2\theta_{\sigma^2}} + \frac{1}{2\theta_{\sigma^2}^2} \sum_{i=1}^{n} \left(x_i - \hat{\theta}_\mu\right)^2 = 0$$

$$\hat{\theta}_{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} \left(x_i - \hat{\theta}_\mu\right)^2$$

In other words, MLE of variance is the *population variance* of the data.



Likelihood – Continuous Case

Definition. The **likelihood** of independent observations $x_1, \dots, x_n$ is

$$\mathcal{L}(x_1, \dots, x_n ; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$$

Normal outcomes $x_1, \dots, x_n$

$$\hat{\theta}_\mu = \frac{\sum_i^n x_i}{n}$$

MLE estimator for **expectation**

$$\hat{\theta}_{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \hat{\theta}_\mu\right)^2$$

MLE estimator for **variance**

General Recipe

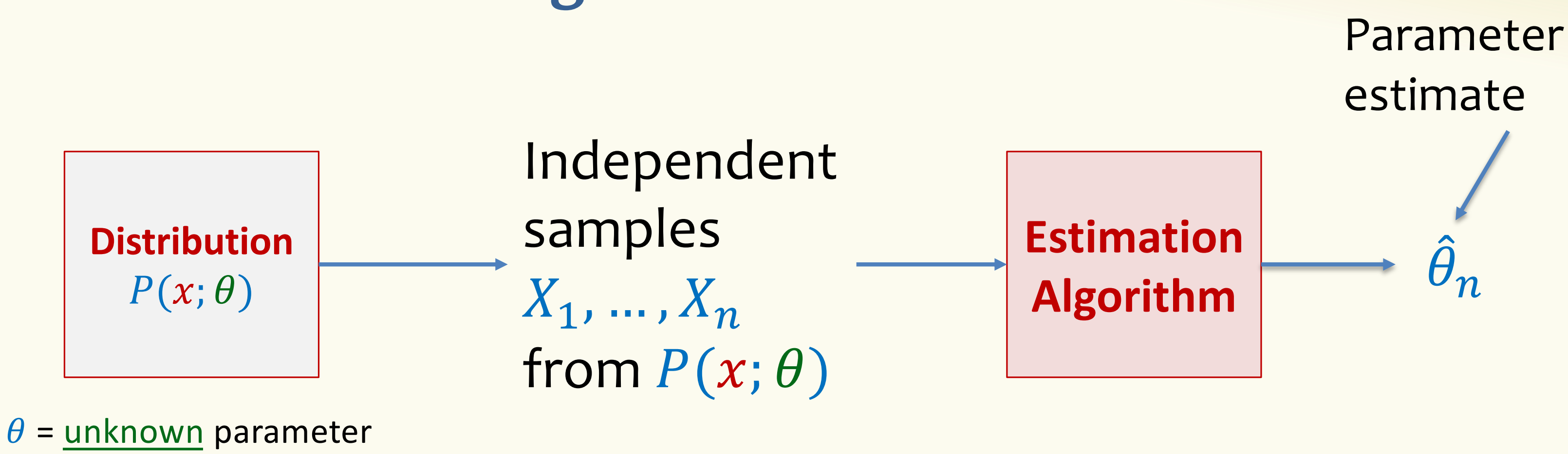
1. **Input** Given n i.i.d. samples $x_1, \ldots, x_n$ from parametric model with parameter θ.
2. **Likelihood** Define your likelihood $\mathcal{L}(x_1, \ldots., x_n | \theta)$.
   - For discrete $\mathcal{L}(x_1, \ldots, x_n ; \theta) = \prod_{i=1}^{n} P(x_i ; \theta)$
   - For continuous $\mathcal{L}(x_1, \ldots, x_n ; \theta) = \prod_{i=1}^{n} f(x_i ; \theta)$
3. **Log** Compute $\ln \mathcal{L}(x_1, \ldots, x_n ; \theta)$
4. **Differentiate** Compute $\frac{\partial}{\partial \theta} \ln \mathcal{L}(x_1, \ldots, x_n ; \theta)$
5. **Solve for** $\hat{\theta}$ by setting derivative to 0 and solving for max.

Generally, you need to do a second derivative test to verify it is a maximum, but we won't ask you to do that in CSE 312.

Agenda

- MLE for Normal Distribution
- Unbiased and Consistent Estimators
- Intuition and Bigger Picture

When is an estimator good?



Definition. An estimator of parameter θ is an **unbiased estimator** if

$$\mathbb{E}[\hat{\theta}_n] = \theta.$$

Note: This expectation is over the samples $X_1, \dots, X_n$

Three samples from $U(0, \theta)$

Example – Coin Flips

Recall: $\hat{\theta}_\mu = \frac{n_H}{n}$

Coin-flip outcomes $x_1, \dots, x_n$, with n_H heads, n_T tails

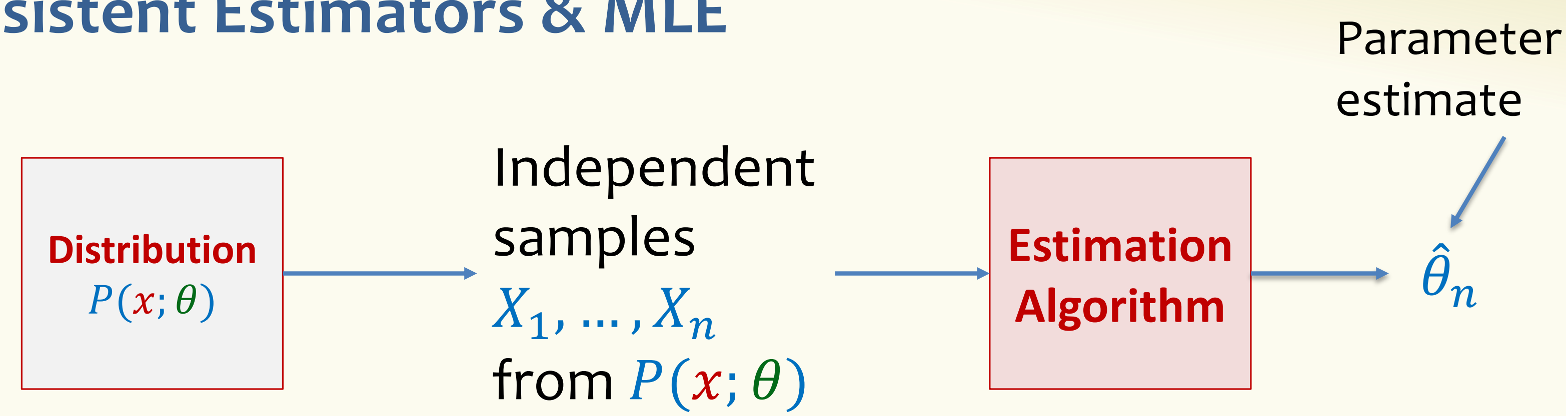
Fact. $\hat{\theta}_\mu$ is unbiased

i.e., $\mathbb{E}[\hat{\theta}_\mu] = p$, where p is the probability that the coin turns out head.

Why?

Because $\mathbb{E}[n_H] = np$ when p is the true probability of heads.

Consistent Estimators & MLE



θ = unknown parameter

Definition. An estimator is **unbiased** if $\mathbb{E}[\hat{\theta}_n] = \theta$ for all $n \geq 1$.

Definition. An estimator is **consistent** if $\lim_{n\to\infty} \mathbb{E}[\hat{\theta}_n] = \theta$.

Theorem. MLE estimators are consistent.

(But not necessarily unbiased)

Example – Consistency

Normal outcomes $X_1, \ldots, X_n$ i.i.d. according to $\mathcal{N}(\mu, \sigma^2)$ Assume: $\sigma^2 > 0$

$$\widehat{\Theta}_{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \widehat{\Theta}_{\mu}\right)^2$$

Population variance – Biased!

$\widehat{\Theta}_{\sigma^2}$ is "consistent"

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Population variance – Biased!

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \widehat{\Theta}_{\mu}\right)^2$$

Sample variance – Unbiased!

$\widehat{\Theta}_{\sigma^2}$ converges to same value as S_n^2, i.e., σ^2, as $n \to \infty$.

$\widehat{\Theta}_{\sigma^2}$ is "consistent"

Why does it matter?

- When statisticians are estimating a variance from a sample, they usually divide by $n-1$ instead of n.
- They and we not only want good estimators (unbiased, consistent)
  - They/we also want **confidence bounds**
    - Upper bounds on the probability that these estimators are far the truth about the underlying distributions
  - Confidence bounds are just like what we wanted for our polling problems, but CLT is usually not the best thing to use to get them (unless the variance is known)