6. Prove that in an (undirected) graph, the relation defined by

$u \sim v$ if and only if there is a simple path between vertices u and v

is transitive.

[Note that the trivial path from a vertex to itself which traverses no edges is a simple path.]

In this problem, you must show that the property of being connected by a simple path is a transitive relation.

Suppose there is a simple path from vertex u to vertex v.

Denote by $x_0, x_1, x_2, \ldots, x_m$ the vertices on such a path with $u = x_0$ and $v = x_m$.

Suppose also that there is a simple path from vertex v to vertex w.

Denote by $y_0, y_1, y_2, \ldots, y_n$ the vertices on such a path with $v = y_0$ and $w = y_n$.

A simple path from u to w must be found.

Let i be the smallest value for which there is a j such that $x_i = y_j$. [In other words, x_i is the first vertex on the simple path from u to v which is also on the simple path from v to w.) Such an i must exist since $x_m = y_0$.

The path $x_0, x_1, x_2, \ldots, x_i, y_{j+1}, y_{j+2}, \ldots, y_n$ is a simple path from u to w since no "x-vertex" is repeated (by definition), no "y-vertex" is repeated (by definition), and no "x-vertex" matches a "y-vertex" (by construction).