

Discrete Structures

Relations

Chapter 6, Sections 6.1 - 6.5

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Relations

- ◇ Let A and B be sets. A **binary relation from A to B** is a subset of $A \times B$. If $(a, b) \in R$, we write aRb and say a is **related to b by R** .
- ◇ A **relation on** the set A is a relation from A to A .
- ◇ A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.
- ◇ A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$, for $a, b \in A$.
- ◇ A relation R on a set A such that $(a, b) \in R$ and $(b, a) \in R$ only if $a = b$, for $a, b \in A$, is called **antisymmetric**.
- ◇ A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for $a, b \in A$.

Examples

$$\diamond R_1 = \{(a, b) \mid a \leq b\}$$

$$\diamond R_2 = \{(a, b) \mid a > b\}$$

$$\diamond R_3 = \{(a, b) \mid a = b \vee a = -b\}$$

$$\diamond R_4 = \{(a, b) \mid a = b\}$$

$$\diamond R_5 = \{(a, b) \mid a = b + 1\}$$

$$\diamond R_6 = \{(a, b) \mid a + b \leq 3\}$$

Combining Relations

- ◇ Let R be a relation from a set A to a set B and S be a relation from B to a set C . The **composite** of R and S is the relation consisting of ordered pairs (a, c) , where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.
We denote the composite of R and S by $S \circ R$.

- ◇ Let R be a relation on the set A . The **powers** R^n , $n = 1, 2, 3, \dots$, are defined inductively by
$$R^1 = R \quad \text{and} \quad R^{n+1} = R^n \circ R.$$

- ◇ **Theorem** : The relation R on a set A is transitive if and only if
$$R^n \subseteq R \text{ for } n = 1, 2, 3, \dots$$

Closures of Relations

- ◇ Let P be a property of relations (transitivity, reflexivity, symmetry). A relation S is **closure of R w.r.t. P** if and only if S has property P , S contains R , and S is a subset of every relation with property P containing R .

Relations and Graphs

- ◇ A **directed graph**, or **digraph**, consists of a set V of **vertices (or nodes)** together with a set E of ordered pairs of elements of V called **edges (or arcs)**.
- ◇ A **path** from a to b in the directed graph G is a sequence of one or more edges $(x_0, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n)$ in G , where $x_0 = a$ and $x_n = b$. This path is denoted by x_0, x_1, \dots, x_n and has **length** n . A path that begins and ends at the same vertex is called a **circuit** or **cycle**.
- ◇ There is a **path** from a to b in a relation R if and only if there is a sequence of elements $a, x_1, x_2, \dots, x_{n-1}, b$ with $(a, x_1) \in R, (x_1, x_2) \in R, \dots, (x_{n-1}, b) \in R$.
- ◇ **Theorem:** Let R be a relation on a set A . There is a path of length n from a to b if and only if $(a, b) \in R^n$.

Connectivity

- ◇ Let R be a relation on a set A . The **connectivity relation** R^* consists of pairs (a, b) such that there is a path between a and b in R .
- ◇ **Theorem:** The transitive closure of a relation R equals the connectivity relation R^* .

Partitions

- ◇ We want to use relations to form partitions of a group of students. Each member of a subgroup is related to all other members of the subgroup, but to none of the members of the other subgroups.

- ◇ Use the following relations:
 - Partition by the relation "older than"
 - Partition by the relation "partners on some project with"
 - Partition by the relation "comes from same hometown as"

- ◇ Which of the groups will succeed in forming a partition? Why?

Equivalence Relations

- ◇ A relation on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive. Two elements that are related by an equivalence relation are called equivalent.

- ◇ Let R be an equivalence relation on a set A . The set of all elements that are related to an element a of A is called the **equivalence class** of a .
 $[a]_R$: equivalence class of a w.r.t. R .
If $b \in [a]_R$ then b is **representative** of this equivalence class.

- ◇ **Theorem:** Let R be an equivalence relation on a set A . The following statements are equivalent:
 - (1) aRb
 - (2) $[a] = [b]$
 - (3) $[a] \cap [b] \neq \emptyset$

Equivalence Relations and Partitions

- ◇ A **partition** of a set S is a collection of disjoint nonempty subsets $A_i, i \in I$ (where I is an index set) of S that have S as their union:

$$A_i \neq \emptyset \text{ for } i \in I$$

$$A_i \cap A_j = \emptyset, \text{ when } i \neq j$$

$$\bigcup_{i \in I} A_i = S$$

- ◇ **Theorem:** Let R be an equivalence relation on a set S . Then the equivalence classes of R form a partition of S . Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S , there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes.