Discrete Structures

Relations

Chapter 6, Sections 6.1 - 6.5

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Relations

◇ Let A and B be sets. A **binary relation from A to B** is a subset of $A \times B$. If $(a, b) \epsilon R$, we write aRb and say a **is related** to b by R.

◇ A **relation on** the set A is a relation from A to A.

◇ A relation R on a set A is called **reflexive** if $(a, a) \epsilon R$ for every element $a \epsilon A$.

◇ A relation R on a set A is called **symmetric** if $(b, a) \epsilon R$ whenever $(a, b) \epsilon R$, for $a, b \ \epsilon \ A$.

◇ A relation R on a set A such that $(a, b) \epsilon R$ and $(b, a) \epsilon R$ only if $a = b$, for $a, b \ \epsilon \ A$, is called **antisymmetric** .

◇ A relation R on a set A is called **transitive** if whenever $(a, b) \epsilon R$ and $(b, c) \epsilon R$, then $(a, c) \epsilon R$, for $a, b \ \epsilon \ A$.

Examples

◇ $R_1 = \{(a, b) \mid a \leq b\}$

◇ $R_2 = \{(a, b) \mid a > b\}$

◇ $R_3 = \{(a, b) \mid a = b \vee a = -b\}$

◇ $R_4 = \{(a, b) \mid a = b\}$

◇ $R_5 = \{(a, b) \mid a = b + 1\}$

◇ $R_6 = \{(a, b) \mid a + b \leq 3\}$

Combining Relations

◇ Let R be a relation from a set A to a set B and S be a relation from B to a set C. The **composite** of R and S is the relation consisting of ordered pairs (a, c), where $a \epsilon A, c \epsilon C$, and for which there exists an element $b \epsilon B$ such that $(a, b) \epsilon R$ and $(b, c) \epsilon S$.
We denote the composite of R and S by $S \circ R$.

◇ Let R be a relation on the set A. The **powers R^n** , $n = 1, 2, 3, \ldots$, are defined inductively by
$R^1 = R$ and $R^{n+1} = R^n \circ R$.

◇ **Theorem** : The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

Closures of Relations

◇ Let P be a property of relations (transitivity, refexivity, symmetry). A relation **S is losure of R w.r.t. P** if and only if S has property P, S contains R, and S is a subset of every relation with property P containing R.

Relations and Graphs

◇ A **directed graph**, or **digraph**, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs).

◇ A **path** from a to b in the directed graph G is a sequence of one or more edges $(x_0, x_1), (x_1, x_2), \ldots (x_{n-1}, x_n)$ in G, where $x_0 = a$ and $x_n = b$. This path is denoted by $x_0, x_1, \ldots, x_n$ and has length n. A path that begins and ends at the same vertex is called a circuit or cycle.

◇ There is a **path** from a to b in a relation R is there is a sequence of elements $a, x_1, x_2, \ldots x_{n-1}, b$ with $(a, x_1) \in R, (x_1, x_2) \in R, \ldots, (x_{n-1}, b) \in R$.

◇ **Theorem:** Let R be a relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

Connectivity

◇ Let R be a relation on a set A. The **connectivity relation R^*** consists of pairs (a, b) such that there is a path between a and b in R.

◇ **Theorem:** The transitive closure of a relation R equals the connectivity relation R^*.

Partitions

◇ We want to use relations to form partitions of a group of students. Each member of a subgroup is related to all other members of the subgroup, but to none of the members of the other subgroups.

◇ Use the following relations:

Partition by the relation ”older than”

Partition by the relation ”partners on some project with”

Partition by the relation ”comes from same hometown as”

◇ Which of the groups will succeed in forming a partition? Why?

Equivalence Relations

- A relation on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive. Two elements that are related by an equivalence relation are called equivalent.

- Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a.
  $[a]_R$: equivalence class of a w.r.t. R.
  If $b \in [a]_R$ then b is representative of this equivalence class.

- **Theorem:** Let R be an equivalence relation on a set A. The following statements are equivalent:

  (1) aRb

  (2) $[a] = [b]$

  (3) $[a] \cap [b] \neq \emptyset$

Equivalence Relations and Partitions

- A **partition** of a set S is a collection of disjoint nonempty subsets $A_i, i \in I$ (where I is an index set) of S that have S as their union:

  $A_i \neq \emptyset$ for $i \in I$

  $A_i \cap A_j = \emptyset$, when $i \neq j$

  $\bigcup_{i \in I} A_i = S$

- **Theorem:** Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes.