Midterm Practice Problems

CSE 321

October 26, 2002

1. If $D(x, y)$ is the predicate "x divides y" then which of the following statements are true in the domain of positive integers?
   (a) $\forall x D(x, x)$.
   (b) $\forall x \forall y (D(x, y) \rightarrow D(y, x))$.
   (c) $\forall x \forall y ((D(x, y) \wedge D(y, x)) \rightarrow (x = y))$.
   (d) $\forall x \forall y (D(x, y) \vee D(y, x))$.
   (e) $\forall x \forall y \forall x ((D(x, y) \wedge D(y, z)) \rightarrow D(x, z))$.

2. Let $P(x, y)$ be the predicate "x is a parent of y", and let $O(x, y)$ be the predicate "x is older than y", and let the universe for all variables be the set of all people. Express each of the following statements as a predicate logic formula using P and O:
   (a) Every parent is older than his/her children.
   (b) Alice and Bob have the same parents.
   (c) John is Mary's oldest child.
   (d) Every person has at least two parents.

   Express this using only universal quantifiers, then using only existential quantifiers: "No two people are exactly the same age."

3. True or false:
   (a) $p \rightarrow q$ is logically equivalent to $\neg p \rightarrow \neg q$.
   (b) $((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$ is a tautology.
   (c) $((\forall x[P(x) \rightarrow Q(x)]) \wedge P(y)) \rightarrow Q(y)$ is a tautology.
   (d) There is a one-to-one function from A to B if and only if there is an onto function from B to A.

(e) To prove by contradiction that $p \to q$, one must show that p is false.

(f) If $A \subseteq B$, then the power set of A is a subset of the power set of B.

(g) If $f : B \to C$ is one-to-one, $f \circ g$ is onto.

(h) If $f : B \to C$ and $g : A \to B$ are both onto, then $f \circ g$ is onto.

4. Prove that gcd(a,b) = gcd(b,a mod b).

5. If a and b are rational numbers, is a^b rational? Prove or disprove this conjecture.

6. Prove by induction that $1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + ... + n \times 2^{n-1} = (n-1) \times 2^n + 1$ whenever n is a positive integer.

7. Consider the following proof:

   Claim : Every natural number is eiter prime or a perfect square.

   Proof : We prove by induction that for all natural numbers n, $P(n)$: n is a prime or a perfect square.

   Base : $P(1)$ is true.

   Inductive hypothesis : Every natural number less than n is prime or a perfect square.

   Inductive step : Consider n. If n is prime, then we are done. Otherwise, n can be factored as $n = rs$ with r and s less than or equal to $n - 1$. By the inductive hypothesis, r and s are perfect squares, so $r = u^2$ and $s = v^2$. Therefore, $n = rs = u^2v^2 = (uv)^2$. So n is a perfect square. Therefore every natural number is prime or a perfect square.

   Which of the following statements are true?

   (a) The proof is wrong because the inductive hypothesis is applied incorrectly. The inductive hypothesis aserts that r and s are either perfect square or ptimes, but the proof uses it to conclude that r and s are perfect squares, ignoring the possibility that they are primes.

   (b) The proof is wrong because it proceeds by trying to prove that n is either a prime or a perfect square. But that is already the inductive hypothesis. Instead the proof should proceed by showing that $n + 1$ is either prime or a perfect square.

(c) the proof is wrong because it is incorrect to claim that "If n is prime then we are done," because this is what we were trying to prove.

(d) The proof is correct.