Practice Questions

from previous CSE321 exams

1. If $D(x, y)$ is the predicate 'x divides y' then which of the following statements are true in the domain of positive integers?
   - $\forall x D(x, x)$.
   - $\forall x \forall y (D(x, y) \rightarrow D(y, x))$.
   - $\forall x \forall y ((D(x, y) \text{AND} D(y, x)) \rightarrow (x = y))$.
   - $\forall x \forall y (D(x, y) \text{OR} D(y, x))$
   - $\forall x \forall y \forall z ((D(x, y) \text{AND} D(y, z)) \rightarrow D(x, z))$.

2. Let $P(x, y)$ be the predicate "x is a parent of y", and let $O(x, y)$ be the predicate "x is older than y", and let the universe for all variables be the set of all people. Express each of the following statements as a predicate logic formula using P and O:
   - Every parent is older than his/her children.
   - Alice and Bob have the same parents.
   - John is Mary's oldest child.
   - Every person has at least two parents.
   - No two people are exactly the same age.

3. True or false:
   - $p \rightarrow q$ is logically equivalent to $q \rightarrow p$.
   - $((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$ is a tautology.
   - $((\forall x[P(x) \rightarrow Q(x)]) \wedge P(y)) \rightarrow Q(y)$ is a tautology.
   - There is a one-to-one function from A to B if and only if there exists an onto function from B to A.
   - To prove by contradiction that $p \rightarrow q$, one must show that p is false.
   - If A is a subset of B, then the power set of A is a subset of the power set of B.
   - For any two sets A and B, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.
   - The rationals are countable.
   - Every subset of a countable set is countable.

   The next two true/false questions refer to functions: $g : A \rightarrow B$ and $f : B \rightarrow C$. (Remember the definition of the composition of two functions: $f \circ g(x) = f(g(x))$.) For each of the following, answer true if the statement is true for *all* f and g satisfying the conditions, and answer false otherwise.

- If f is one-to-one, $f \circ g$ is one-to-one.

- If f and g are both onto, then $f \circ g$ is onto.

4. Find $\gcd(2n+1, 3n+2)$, where n is a positive integer. Hint: Use the Euclidean algorithm.

5. Define a function g on the non-negative integers by $g(0) = 2$, $g(1) = 3$ and $g(n+1) = 3g(n) - 2g(n-1)$ for all $n \geq 1$. Use strong induction to prove that for all $n \geq 0$, $g(n) = 2^n + 1$.

6. Prove by contradiction: If x is a rational number and y is an irrational number, then $x + y$ is irrational. You may use the following fact: if a and b are nonzero rational numbers, then a/b is also rational.

7. Prove by induction that

$$1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \ldots + n \cdot 2^{n-1} = (n-1) \cdot 2^n + 1$$

whenever n is a positive integer.

8. Consider the following proof:

Claim: *Every natural number is either prime or a perfect square.*[1]

Proof: *We prove by induction that for all natural numbers n, $P(n)$: n is a prime or a perfect square.*
Base case: *$P(1)$ is certainly true.*
Inductive Hypothesis: *Every natural number less than n is a prime or a perfect square.*
Inductive step: *Consider n. If n is prime, then we are done. Otherwise, n can be factored as $n = rs$ with r and s less than or equal to $n-1$. By the inductive hypothesis, r and s are perfect squares, so $r = u^2$ and $s = v^2$. Therefore, $n = rs = u^2v^2 = (uv)^2$. So n is a perfect square. Therefore every natural number is either prime or a perfect square.*

- (4 points) Does this proof use regular induction or strong induction?

- (8 points) Which of the following statements are true? (There could be zero, one, or more than one true statement.)
  (a) The proof is wrong because the inductive hypothesis is applied incorrectly. The inductive hypothesis asserts that r and s are **either** perfect squares **or** primes, but the proof uses it to conclude that r and s are perfect squares, ignoring the possibility that they are primes.
  (b) The proof is wrong because it proceeds by trying to prove that n is either a prime or a perfect square. But that is already the inductive hypothesis. Instead the proof should proceed by showing that $n+1$ is either prime or a perfect square.
  (c) The proof is wrong because it is incorrect to claim that "If n is prime, then we are done," because this is what we were trying to prove.
  (d) The proof is correct.

[1] An integer a is a *perfect square* if there is an integer b such that $a = b^2$.