CSE321 Exam 2 Review Sample Problem Solutions
June 5, 2003

Sample Problems:

1. Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps:

   (a) by strong induction.

   Base cases: 12,13,14,and 15 cents can be made from 4- and 5-cent stamps.

   Inductive hypothesis: We can make i cents of postage from 4- and 5-cent stamps if $12 \leq i < n$.

   To prove: We can make n cents of postage using 4- and 5-cent stamps.

   Inductive step ($n > 15$): We can make n cents of postage by making $n - 4$ cents of postage and adding a 4-cent stamp.

   (b) by weak induction.

   Base cases: 12 cents can be made from 4- and 5-cent stamps.

   Inductive hypothesis: We can make n cents of postage from 4- and 5-cent stamps.

   To prove: We can make $n + 1$ cents of postage using 4- and 5-cent stamps.

   Inductive step ($n \geq 15$): We have two cases for this proof:

   i. We used at least one 4-cent stamp to make n cents of postage:
   In this case we can take out the 4-cent stamp and replace it with a 5-cent stamp, making $n + 1$ cents of postage.

   ii. We used at least three 5-cent stamps to make n cents of postage:
   In this case we can remove three of the 5-cent stamps and add four 4-cent stamps, making $n + 1$ cents of postage.

   If we don't use one 4-cent stamp to make n cents, then we use at least three 5-cent stamps, since $n > 10$. Our two cases are sufficient.

2. This question deals with the probability of choosing a random string of 10 bits having a substring of at least 5 consecutive zeros.

   (a) Why is the probability not equal to the number of places to put a string of 5 zeros times the number of for the other bits, divided by the total number of 10 bit strings, or $6 * 2^5/2^{10}$?

   We overcount cases where we have greater than 5 consecutive zeros. 0000000000, for example, is counted 6 times.

   (b) What is the probability? (This is a harder question than what you will see on the final.)

   We can eliminate the possibility of repeating the string of 5 zeros by adding a 1 to the front of it. We can count the number of strings with 100000 as a substring by the method used above, giving $5 * 2^4$. We then need to add in the number of strings that start with 00000, or 2^5. Our probability is then $(2^5 + 5 * 2^4)/2^{10}$.

Note: The include/exclude method that was previously here was wrong. It should be just $6 * 2^5 - 5 * 2^4$.

3. We define a relation R over a graph $G = (V, E)$ as uRv iff. $u, v \in V$ and there is a path in G from u to v.

   (a) Is R a reflexive, symmetric, and/or transitive relation if G is an arbitrary undirected graph?

   R is reflexive: A node is reachable from itself.
   R is symmetric: Edges go both ways, so if u is reachable from v then v is reachable from u.
   R is transitive: If we can reach v from u, and w from v, then we can combine the path to reach w from u.

   (b) Is R a reflexive, symmetric, and/or transitive relation if G is an arbitrary directed graph?

   R is reflexive: A node is reachable from itself.
   R is not necessarily symmetric: The graph with 2 vertices and one edge makes R not symmetric.
   R is transitive: If we can reach v from u, and w from v, then we can combine the path to reach w from u.

   (c) Do equivalence classes exist for (a) and (b), and if so describe them.

   The equivalence classes for R on an undirected graph are the connected components of that graph. R is not guaranteed to be symmetric on an undirected graph, so it doesn't necessarily have equivalence classes.

4. Suppose that a connected bipartite planar simple graph has e edges and v vertices. Show that $e \leq 2v - 4$ if $v \geq 3$. Use this to show that $K_{3,3}$ is not planar.

   We use the notion of the degree of a region given on page 505. Since we have a bipartite graph, the regions created must have degree greater than or equal to 4, since we need at least 3 edges to form a region and we can have no circuits with an odd number of edges in a bipartite graph. If we have r regions, e edges and v vertices, then
   $2e = \sum_{allregionsR} \deg(R) \geq 4r$
   Hence, $(1/2)e \geq r$. Using Euler's formula ($r = e - v + 2$), we obtain
   $e - v + 2 \leq (1/2)e$
   It follows that $e/2 \leq v - 2$, or $e \leq 2v - 4$

   The number of edges in $K_{3,3}$ is 9, which is more than $2v - 4$, or 8. Therefore $K_{3,3}$ is not a planar graph.

5. Give an example of a relation that is:

(a) symmetric and antisymmetric
    aRb iff. $a = b$.

(b) neither symmetric nor antisymmetric
    $S = \{a, b, c\}$
    $R = \{(a, b), (b, a), (a, c)\}$

6. A chip has 5 identical components each with 20% failure rate. The chip fails if at least 2 components fail. What is probability that the chip fails?

   We can subtract the probability that 1 or 0 components fail from 1. The probability that no components fail is $.8^5$ The probability that one component fails is $5 * (.2)(.8)^4$ This gives us a failure probability of about 26%.