CSE 321: Discrete Structures
Assignment #6
November 12, 2004
Due: Friday, November 19

Reading Assignment: Read Sections 5.1, 5.2, 5.3 (only the part on expectation).

Problems:

1. Section 4.4, exercise 8.
2. Section 4.4, exercise 28.
3. An ice cream parlor has 28 different flavors, 8 different kinds of sauce, and 12 toppings.
   (a) In how many different ways can a dish of three scoops of ice cream be made where each flavor can be used more than once and the order of the scoops does not matter?
   (b) How many different kinds of small sundaes are there if a small sundae contains one scoop of ice cream, a sauce, and a topping?
   (c) How many different kinds of large sundaes are there if a large sundae contains three scoops of ice cream, where each flavor can be used more than once and the order of the scoops does not matter; two kinds of sauce, where each sauce can be used only once and the order of the sauces does not matter; and three toppings, where each topping can be used only once and the order ot toppings does not matter?
4. Section 5.1, exercise 18.
5. Section 5.1, exercise 28.
6. Section 5.1, exercise 36 (a total of exactly 8 is meant). Justify your answer.
7. A deck of 10 cards, each bearing a distinct number from 1 to 10, is shuffled to mix the cards thoroughly, so that each order is equally likely. What is the probability that the top three cards are in sorted (increasing) order?
8. A fair coin is flipped n times. What is the probability that all the heads occur at the end of the sequence?
9. **Extra Credit: Due First Week of December** In cryptography, one typically needs to choose random primes of a certain size. In order to do this people simply choose random numbers and then check to see if they are prime. In order for this to work efficiently, the number of primes has to be plentiful. The following sequence of problems will direct you to produce a proof that primes are indeed plentiful. (The exact answer is closely related to the Reimann Hypothesis whose solution is worth a $1 million prize.)
   (a) Show that for any prime p, the largest power of p that divides $n!$ is

   $$\lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \cdots + \lfloor \frac{n}{p^r} \rfloor$$

   where $p^r \leq n < p^{r+1}$.

(b) Use the basic definition (no induction) to show that for any $m \geq 1$, $\lfloor \frac{2n}{m} \rfloor \leq 2\lfloor \frac{n}{m} \rfloor + 1$.

(c) Use the formula for $\binom{2n}{n}$ and the results of parts (a) and (b) to show that for any prime p, the largest power p^r of p that divides $\binom{2n}{n}$ satisfies $p^r \leq 2n$.

(d) Prove that for any integer $n \geq 1$, $\binom{2n}{n} \geq 2^n$.

(e) Use the lower bound on the size of $\binom{2n}{n}$ from part (d) and upper bound on each of its prime power factors from part (c) to prove that the number of distinct primes dividing $\binom{2n}{n}$ is at least $n/\log_2(2n)$.

(f) Conclude that there are at least $n/\log_2(2n)$ primes less than $2n$.

NOTE: You can use the results of previous parts to solve later parts even if you haven't finished the earlier parts.