CSE 321: Discrete Structures

Solution to Practice Mid-term:

1. (a)

   - $\forall x \forall y \forall z [D(x,y) \rightarrow D(x,yz)]$.
     It's correct. Since when x is a divisor of y, x is also a divisor of yz.
   - $\forall x \forall y \forall z \forall w [D(xy,zw) \rightarrow (D(x,z) \vee D(y,w))]$.
     It's not correct. For example, $x = 3, y = 8, z = 4, w = 6$.

   (b) Denote "$a > b$" by $E(a,b)$, and "$a \neq b$" by $F(a,b)$.

   - "p is a prime number"
     $\forall x [D(x,p) \rightarrow (\neg F(x,p) \vee \neg F(x,1))]$
   - "q has at least two positive divisors greater than 1"
     $\exists x [(D(x,q) \wedge E(x,1) \wedge E(q,x)]$

2. (a) True

   (b) True

   (c) True. Since q is independent to x.

   (d) True

   (e) True

   (f) True.

   (g) True

   (h) False

   (i) False

   (j) True

   (k) False. For example, $A = \{1,2\}, B = \{1,2,3\}$, $f(x) = x$ and $g(x) = 3 - x$.

   (l) True

   (m) True

3. By mathematical induction.

   Basis step: n=1. We have $1 \cdot 2^0 = 1 = (1-1) \cdot 2^1 + 1$.

   Inductive hypothesis: Assume the argument is true for $n = k$, *i.e.*,

$$1 \cdot 2^0 + \cdots + k \cdot 2^{k-1} = (k-1) \cdot 2^k + 1.$$

   Inductive step: When $n = k+1$, we have

$$\begin{aligned} 1 \cdot 2^0 + \cdots + k \cdot 2^{k-1} + (k+1) \cdot 2^k &= (k-1) \cdot 2^k + 1 + (k+1) \cdot 2^k \\ &= 2k \cdot 2^k - 2^k + 1 + 2^k \\ &= k \cdot 2^{k+1} + 1 \end{aligned}$$

   which completes the proof.

4. (i) It uses strong induction. Since in the inductive hypothesis, we assume each natural number less than n is a prime or a perfect square.

   (i) Only (a) is correct. The reason why (b) is not correct is that our induction hypothesis is w.r.t all natural numbers less than n, and we only need to prove the case of n in the inductive step. The reason why (c) is not correct is that "If n is prime, then we are done" is one of two cases of our proof, and if this holds, we are done.