CSE 321: Discrete Structures — Assignment #7

November 22, 2006 — Due: Friday, December 1

Reading Assignment: Rosen's text 6th Edition: finish sections 6.3-6.4 (to page 433) and read sections 8.1-8.4 or 5th Edition: sections 7.1-7.4.

Problems:

1. 6th Edition Section 6.3, Exercise 4. (5th Edition Section 5.3, Exercise 4.)

2. In a player's turn during the game of Monopoly, the player will roll a pair of dice to determine the number of steps they will move (which is the sum of the values on the two dice). If the player rolls a double (the two numbers match) then they get to take another turn immediately, otherwise the next player gets to move. The same thing happens if they roll doubles a second time. If they roll three doubles in a row they must move directly to "jail" and the next player gets to move. (In answering the following questions please ignore the fact that in the real game of Monopoly, by moving to certain locations, one may be sent to other locations on the board or one's turns may be halted prematurely by being bankrupted.)

   (a) What is the probability that they will roll a double on their first turn?

   (b) What is the expected number of steps they will move on their first turn, given that they rolled doubles? given that they didn't roll doubles?

   (c) What is the probability that they will roll doubles at least twice in a row?

   (d) What is the probability that they will roll doubles all three times and therefore be sent to jail by their third roll?

   (e) Suppose that the rules were changed so that players simply stopped rolling after their second double. What would be the expected total number of steps they would have moved by the time the next player got to move if we made this change to the rules?

   (f) **Extra credit**: What is the expected number of steps they will move given that they do not get sent to jail by rolling doubles three times in a row? (In other words, as long as you keep rolling doubles, you keep getting another turn.)

3. In the following problems, you are given a 5-card hand from a randomly shuffled deck of 52 cards.

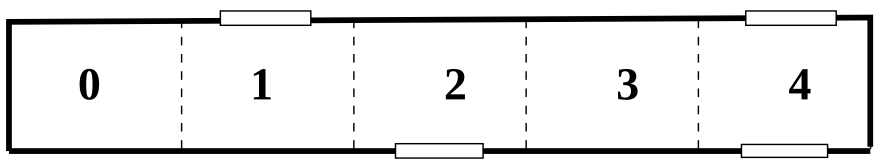
   (a) Given that you have at least one ace, what's the probability that you have another ace?

   (b) Given that you have the ace of diamonds, what's the probability that you have another ace?

(c) Given that you have a red ace (diamonds and hearts are red), what's the probability that you have another ace?

(d) Suppose you select a random card from your hand and it's an ace. What's the probability that you have another ace?

4. *Mobile robot localization*: Bayes' rule underlies all modern AI systems for probabilistic inference (see also Rosen, page 397, exercises 25 and 26). One application of this rule is the update of a robot's position estimate based on new sensor information. To see, look at the example below.



The robot is placed in the hallway facing east and it does not know where it is (it only knows its orientation). The hallway is tessellated and the robot can be in any of the five squares (always facing east). Assume that the width of each square is 1 meter. Then the squares are numbered by their distance from the beginning of the hallway. Given the tessellation of the hallway, we can represent the position of the robot by a random variable L. This variable takes on values between 0 and 4, depending on the robot's current position.

At each point in time, the robot's position estimate is represented by a probability distribution over all possible locations. In the beginning, the robot does not know where it is. This can be represented by the following distribution:

$$P(L = 0) = P(L = 1) = P(L = 2) = P(L = 3) = P(L = 4) = 1/5 \tag{1}$$

The robot has a camera that can be pointed to the left or to the right. In order to determine where it is, the robot tries to detect doors by looking either to the left or to the right (note that the robot faces east). As you can see in the figure, there are four doors in the hallway.

Now we want to update the robot's position estimate based on an observation, i.e. we want to compute the probability distribution of L *given* an observation. In general, we denote observations by o (where o ranges over the possible observations). The probability distribution *after* the observation can be represented by $P(L = l|o)$ (l is a number between 0 and 4). However, it is not easy to compute this value directly. In mobile robot localization we use Bayes' rule to compute this value:

$$P(L = l|o) \quad = \quad \frac{P(o \mid L = l)P(L = l)}{\sum_{i=1}^{n} P(o \mid L = l_i)P(L = l_i)} \tag{2}$$

All we need to know are the values for $P(o \mid L = l)$ and $P(L = l)$. $P(L = l)$ is given by the distribution *before* the robot made the observation (given in Equation (1)).

The term $P(o \mid L = l)$ describes the probability of making an observation *given* the position of the robot. In our example, we have four possible observations: The robot can detect a door to its left, it can detect a door to its right, it can detect a wall to its left, or it can detect a wall to its right. We will denote these four observations by DL, DR, WL, and WR (i.e. these are the four possible values of o). To determine $P(o \mid L = l)$, the probability of making observation o at location l, we make the following assumption about the robot's ability to detect doors and walls. If the robot points the camera towards a door, then it successfully detects this door with probability 0.7. However, it sometimes confuses the door for a wall, i.e. with probability 0.3 it erroneously detects a wall, even though it looks at a door. Detecting walls is easier, therefore, the probability of detecting a wall if the robot is looking at a wall is 0.9. With probability 0.1 it thinks it detects a door even though it looks at a wall. There are four different types of positions in the hallway: Let l_{DL} denote any position in the hallway, at which there is a door to the left of the robot (these are positions 1 and 4 in our example). The other three types are given by l_{DR},l_{WL} and l_{WR}. For example, l_{WR} are positions 0, 1 and 3. Now we can summarize the robot's ability to detect doors and walls as follows:

$$
\begin{aligned}
P(DL|L = l_{DL}) &= 0.7 \\
P(DL|L = l_{WL}) &= 0.1 \\
P(WL|L = l_{WL}) &= 0.9 \\
P(WL|L = l_{DL}) &= 0.3 \\
P(DR|L = l_{DR}) &= 0.7 \\
P(DR|L = l_{WR}) &= 0.1 \\
P(WR|L = l_{WR}) &= 0.9 \\
P(WR|L = l_{DR}) &= 0.3
\end{aligned}
$$

For example, the probability that the robot detects a door to its left if it is at location 3, is determined by $P(DL|L = l_{WL})$, since there is actually a wall to the robot's left side at position 3. Now we can start to estimate the position of the robot.

(a) Assume that the robot does not know where it is ($P(L)$ is given by equation (1)). The robot detects a wall to it's right, *i.e.* it makes the observation WR. What is the distribution of the robot's position estimate after this observation? Use a calculator (or write a computer program) and Bayes' rule.

(b) Assume the robot has already updated its position estimate based on the first observation WR. Now the robot turns the camera and looks to the left. What is the robot's position estimate, if it detects a door to the left? Use the distribution you computed in the previous step to represent $P(L = l_i)$. This probability distribution combines the fact that the robot has detected a wall to its right and a door to its left.

(c) Compute the mean of the two previous distributions over possible robot positions.

(d) After these two observations, the robot points the camera back to the right and, oops, it detects a door! Compute the new probability distribution and briefly discuss whether the new values make sense.

(e) We might have assumed that the robot never fails when it detects a door or a wall. How could you represent such a perfect sensor model using the conditional probabilities? What would the distributions look like after the observations WR, DL and DR?

5. 6th edition Section 8.1, Exercise 6. 5th edition Section 7.1, Exercise 6.

6. **Extra Credit:** The 120 seats on a Northeast Airlines flight were completely booked, with each of the 120 passengers having different assigned seats. The passengers entered the plane one-by-one. Unfortunately, the first passenger couldn't read their boarding pass and sat in a (uniformly) random seat. Each subsequent passenger sat in their assigned seat if it was available when they entered and sat in a (uniformly) random empty seat otherwise. What is the probability that the last passenger sat in their assigned seat?

7. **Extra Credit:** In cryptography, one typically needs to choose random primes of a certain size. In order to do this people simply choose random numbers and then check to see if they are prime. In order for this to work efficiently, the number of primes has to be plentiful. The following sequence of problems will direct you to produce a proof that primes are indeed plentiful. (The exact answer is closely related to the Reimann Hypothesis whose solution is worth a \$1 million prize.)

(a) Show that for any prime p, the largest power of p that divides $n!$ is

$$\lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \cdots + \lfloor \frac{n}{p^r} \rfloor$$

where $p^r \leq n < p^{r+1}$.

(b) Use the basic definition (no induction) to show that for any $m \geq 1$, $\lfloor \frac{2n}{m} \rfloor \leq 2\lfloor \frac{n}{m} \rfloor + 1$.

(c) Use the formula for $\binom{2n}{n}$ and the results of parts (a) and (b) to show that for any prime p, the largest power p^r of p that divides $\binom{2n}{n}$ satisfies $p^r \leq 2n$.

(d) Prove that for any integer $n \geq 1$, $\binom{2n}{n} \geq 2^n$.

(e) Use the lower bound on the size of $\binom{2n}{n}$ from part (d) and upper bound on each of its prime power factors from part (c) to prove that the number of distinct primes dividing $\binom{2n}{n}$ is at least $n/\log_2(2n)$.

(f) Conclude that there are at least $n/\log_2(2n)$ primes less than $2n$.

NOTE: You can use the results of previous parts to solve later parts even if you haven't finished the earlier parts.