CSE 321: Discrete Structures — Assignment #4

October 19, 2007 — Due: **Monday**, October 29

Reading Assignment: 6th Edition: 3.5–3.6 and 4.1–4.2 (or, 5th Edition: 2.4–2.5 & 3.3).

Problems:

1. Prove that if n is an integer, then $n^2 \bmod 8$ is either 0, 1, or 4.

2. Compute the greatest common divisor for each of the following pairs of numbers.

   (a) $2^2 \cdot 3^3 \cdot 5^5, 2^5 \cdot 3^3 \cdot 5^4$

   (b) $1000, 625$

   (c) $20!, 127$

3. Prove that if a, b, and m are integers such that $m \geq 2$ and $a \equiv b \pmod{m}$ then $\gcd(a, m) = \gcd(b, m)$.

4. Use induction to show that $1^3 + 2^3 + \cdots + n^3 = [n(n+1)/2]^2$ whenever n is a positive integer.

5. Casting out nines: Prove that a number is divisible by 9 if and only if the sum of its digits is divisible by 9. More specifically, for a positive integer a, let $\text{digitsum}(a)$ be the sum of the decimal digits of a. Prove by induction on the number of decimal digits in a that $a \equiv \text{digitsum}(a) \pmod{9}$.
   Hint: Express a as $\sum_{i=0}^{n} a_i(10)^i$ where $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

6. Sometimes it's easier to prove a stronger statement than is apparently required. In this problem you will prove by induction that for all $n \geq 1$,

   $$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} \leq \frac{3}{2}.$$

   (a) What doesn't work if you try to produce an inductive proof in which $P(n)$ is the statement that $\sum_{k=1}^{n} \frac{1}{k^3} \leq \frac{3}{2}$?

   (b) Now use induction to prove the stronger statement that for all $n \geq 1$, $\sum_{k=1}^{n} \frac{1}{k^3} \leq \frac{3}{2} - \frac{1}{2n^2}$.

7. Find the flaw with the following "proof" that $a^n = 1$ for all non-negative integers n, whenever $a > 0$.

   (a) **Basis step:** $a^0 = 1$ is true by the definition of a^0.

   (b) **Inductive step:**

   $$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1.$$

EXTRA CREDIT ON BACK

8. **Extra credit:** Use the results of problem 1 above to show that the equation

$$3x^2 - 2y^2 = 69$$

does not have any solution with x and y both integers.

9. **Extra credit:** Consider any $n + 1$ numbers between 1 and $2n$ (inclusive). Show that some pair of them is relatively prime. Show that one is a factor of another.