CSE 321: Discrete Structures
November 20th, 2007

Assignment #7
Due: **Friday**, November 30th

Reading Assignment: 6th Edition: 6.1-6.4 (or, 5th Edition: 5.1–5.4).

Problems:

1. What is the probability that a five card poker hand contains exactly two hearts and two diamonds? Assume that a 5-card hand is dealt from a randomly shuffled deck of 52 cards.

2. Suppose we choose randomly and independently two subsets A and B from the set of all possible non-empty subsets of $\{1, 2, \ldots, n\}$. What is the probability that $\min(A) = \min(B)$ (where $\min(A)$ denotes the minimum number from the set A)?

3. A fair coin is flipped n times. What is the probability that all the heads occur at the start of the sequence?

4. Suppose a 6-sided fair dice is rolled. Let the random variable X be the value showing. What is the expectation of X? Suppose two fair 6-sided dice are rolled independently. Let Y be the random variable which is the sum of the two values showing. What is the expectation of Y ? Let Z be the random variable which is the minimum of the two values showing. What is the expected value of Z?

5. In the following problems, you are given a 5-card hand from a randomly shuffled deck of 52 cards.

   (a) Given that you have at least one ace, whats the probability that you have another ace?

   (b) Given that you have the ace of diamonds, whats the probability that you have another ace?

   (c) Given that you have a red ace (diamonds and hearts are red), whats the probability that you have another ace?

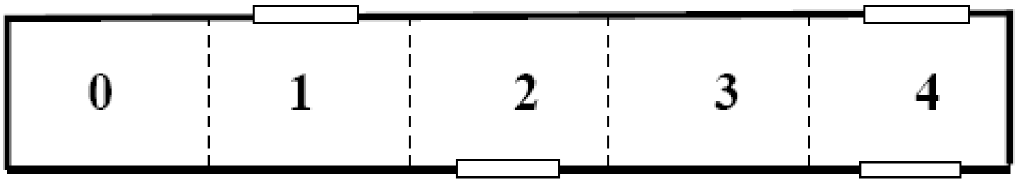
   (d) Suppose you select a random card from your hand and its an ace. Whats the probability that you have another ace?

6. (a) Prove or disprove: If events E and F are independent, then so are $\bar{E}$ and $\bar{F}$.

   (b) Let A, B be events from a sample space with $Pr(A) \neq 0$ and $Pr(B) \neq 0$. Prove or disprove: If A makes B less likely then B makes A less likely. In other words prove or disprove: if $Pr(B|A) < Pr(B)$, then $Pr(A|B) < Pr(A)$.

   (c) Prove or disprove: For a random variable X, $E(X^2) = E(X)^2$.

7. *Mobile robot localization:* Bayes rule underlies all modern AI systems for probabilistic inference. One application of this rule is the update of a robots position estimate based on new sensor information. To see, look at the example below.



The robot is placed in the hallway facing east and it does not know where it is (it only knows its orientation). The hallway is tessellated and the robot can be in any of the five squares (always facing east). Assume that the width of each square is 1 meter. Then the squares are numbered by their distance from the beginning of the hallway. Given the tessellation of the hallway, we can represent the position of the robot by a random variable L. This variable takes on values between 0 and 4, depending on the robots current position. At each point in time, the robots position estimate is represented by a probability distribution over all possible locations. In the beginning, the robot does not know where it is. This can be represented by the following distribution:

$$Pr(L = 0) = Pr(L = 1) = Pr(L = 2) = Pr(L = 3) = Pr(L = 4) = \frac{1}{5}. \tag{1}$$

The robot has a camera that can be pointed to the left or to the right. In order to determine where it is, the robot tries to detect doors by looking either to the left or to the right (note that the robot faces east). As you can see in the figure, there are four doors in the hallway.

Now we want to update the robots position estimate based on an observation, i.e. we want to compute the probability distribution of L *given* an observation. In general, we denote observations by o (where o ranges over the possible observations). The probability distribution after the observation can be represented by $Pr(L = l|o)$ (l is a number between 0 and 4). However, it is not easy to compute this value directly. In mobile robot localization we use Bayes rule to compute this value:

$$Pr(L = l|o) = \frac{Pr(o|L = l)Pr(L = l)}{\sum_{i=1}^{n} Pr(o|L = l_i)Pr(L = l_i)}$$

All we need to know are the values for $Pr(o|L = l)$ and $Pr(L = l)$. $Pr(L = l)$ is given by the distribution before the robot made the observation (given in Equation (1)). The term $Pr(o|L = l)$ describes the probability of making an observation given the position of the robot. In our example, we have four possible observations: The robot can detect a door to its left, it can detect a door to its right, it can detect a wall to its left, or it can detect a wall to its right. We will denote these four observations by DL, DR, WL, and WR (i.e. these are the four possible values of o). To determine $Pr(o|L = l)$, the probability of making observation o at location l, we make the following assumption about the robots ability to detect doors and walls. If the robot points the camera towards a door, then it successfully detects this door with probability 0.7. However, it sometimes confuses the door for a wall, i.e. with probability 0.3 it erroneously detects a wall, even though it looks at a door. Detecting walls is easier, therefore, the probability of detecting a wall if the robot is looking at a wall is 0.9. With probability 0.1 it thinks it detects a door even though it looks at a wall. There are four different types of positions in the hallway: Let l_{DL} denote any position in the hallway, at which there is a door to the left of the robot (these are positions 1 and 4 in our example). The other three types are given by l_{DR}, l_{WL} and l_{WR}. For example, l_{WR} are positions 0, 1 and 3. Now we can summarize the robots ability to detect doors and walls as follows:

$$\begin{aligned}
Pr(DL|L = l_{DL}) &= 0.7 \\
Pr(DL|L = l_{WL}) &= 0.1 \\
Pr(WL|L = l_{WL}) &= 0.9 \\
Pr(WL|L = l_{DL}) &= 0.3 \\
Pr(DR|L = l_{DR}) &= 0.7 \\
Pr(DR|L = l_{WR}) &= 0.1 \\
Pr(WR|L = l_{WR}) &= 0.9 \\
Pr(WR|L = l_{DR}) &= 0.3
\end{aligned}$$

For example, the probability that the robot detects a door to its left if it is at location 3, is determined by $Pr(DL|L = l_{WL})$, since there is actually a wall to the robots left side at position 3. Now we can start to estimate the position of the robot.

(a) Assume that the robot does not know where it is ($Pr(L)$ is given by equation (1)). The robot detects a wall to its right, i.e. it makes the observation WR. What is the distribution of the robots position estimate after this observation? Use a calculator (or write a computer program) and Bayes' rule.

(b) Assume the robot has already updated its position estimate based on the first observation WR. Now the robot turns the camera and looks to the left. What is the robots position estimate, if it detects a door to the left? Use the distribution you computed in the previous step to represent $Pr(L = l_i)$. This probability distribution combines the fact that the robot has detected a wall to its right and a door to its left.

(c) After these two observations, the robot points the camera back to the right and, oops, it detects a door! Compute the new probability distribution and briefly discuss whether the new values make sense.

8. **Extra Credit**: The 120 seats on an Alaska Airlines flight were completely booked, with each of the 120 passengers having different assigned seats. The passengers entered the plane one-by-one. Unfortunately, the first passenger, Joe Isclumsy, couldn't read his boarding pass because he spilled coffee on it and sat in a (uniformly) random seat. Each subsequent passenger sat in their assigned seat if it was available when they entered and sat in a (uniformly) random empty seat otherwise. What is the probability that the last passenger sat in their assigned seat?