Relations

Definition Let A and B be sets. A *binary relation* R from A to B is a subset of $A \times B$. (In this class, we will only concern ourselves with binary relations.)

Example:

Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$, then any subset of

$$A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z), (3, x), (3, y), (3, z), (4, x), (4, y), (4, z)\}$$

is a relation from A to B.

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Let $a \in A$, $b \in B$, and R be a relation from A to B, then:

$a \; R \; b$ denotes that $(a, b) \in R$. a is **related to** b.

$a \not R \; b$ denotes that $(a, b) \notin R$

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Definition A *relation on the set* A is a relation from A to A.

Example:

$R = \{(x, y) \mid x < y\}$ is a relation on the set of integers if $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$.

Notice that relations need not be finite. The relation R above is infinite. There are an infinite number of tuples (x, y) such that $x < y$.

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The following properties (reflexive, symmetric, antisymmetric, transitive) are used to classify relations on a set.

Definition A relation R on a set A is *reflexive* if $(a, a) \in R$ for every element $a \in A$. In other words, R is reflexive if $\forall a \in A((a, a) \in R)$.

Example:

Let $A = \{1, 2, 3, 4\}$.

Any relation on the set A that is reflexive must have the tuples $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$. It can have other tuples, but to be reflexive, it must have those four tuples at the minimum.

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Definition A relation R on the set A is *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$. In other words, $\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$ for $a, b \in A$.

Example:

Let R be a relation on the set of integers.

$R = \{(x, y) \mid x < y\}$ is not symmetric, because $(1, 2) \in R$, but $(2, 1) \notin R$.

$R = \{(x, y) \mid x \neq y\}$ is symmetric, because for every $(x, y) \in R$, (y, x) is also in R.

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Definition A relation R on a set A is *antisymmetric* if it has the following property: For all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$. In other words, $\forall a \forall b(((a, b) \in R \wedge (b, a) \in R) \rightarrow a = b)$.

Example:

$R = \{(x, y) \mid x < y\}$ is antisymmetric, because for every $(x, y) \in R$, (y, x) is NOT in R.

$R = \{(x, y) \mid x \neq y\}$ is not antisymmetric, because $(1, 2) \in R$, but so is $(2, 1)$, but $1 \neq 2$ (making the conclusion to the implication false).

Note: Even though the examples shown seem to indicate that being *symmetric* and *antisymmetric* are opposites. They are actually not opposites. One of your homework problems asks you to come up with a relation that is both and another that is neither.

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Definition A relation R on a set A is *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for all $a, b, c \in A$. In other words, $\forall a \forall b \forall c(((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R)$.

Example:

$R = \{(x, y) \mid x < y\}$ is transitive, because if $x < y$ and $y < z$, then $x < z$ and thus $(x, z) \in R$.

$R = \{(x, y) \mid x + y = 3\}$ is not transitive, because $(1, 2) \in R$ and $(2, 1) \in R$, but $(1, 1) \notin R$.

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As relations are sets, you can use any set operator to combine them.

Example:

Let $R_1 = \{(x, y) \mid x < y\}$ and $R_2 = \{(x, y) \mid x > y\}$.

$R_1 \cup R_2 = \{(x, y) \mid x \neq y\}$

$R_1 \cap R_2 = \emptyset$

$R_1 - R_2 = R_1$ (R_1 and R_2 are disjoint, so subtracting R_2 from R_1 does nothing)

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Definition Let R be a relation from a set A to a set B and S be a relation from B to a set C. The *composite* of R and S is the relation consisting of the ordered pairs (a, c) where $a \in A, c \in C$ for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. The composite of R and S is denoted by $S \circ R$. In other words, $S \circ R = \{(a, c) \mid \exists b((a, b) \in R \wedge (b, c) \in S)\}$.

Example:

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$ and $C = \{x, y, z, w\}$.

Suppose $R = \{(1, a), (1, c), (2, b), (4, b), (4, c)\}$ and $S = \{(a, x), (a, w), (b, y), (b, z), (b, w), (c, w)\}$.

Then $S \circ R = \{(1, x), (1, w), (2, y), (2, z), (2, w), (4, y), (4, z), (4, w)\}$.

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You should notice the similarity between the definition of the composite of two relations and a relation being transitive. First, let's define the composite of relations on a set.

Definition Let R be a relation on a set A. The powers R^n, $n = 1, 2, 3, ...$, are defined recursively by:

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R.$$

Theorem Let R be a relation on a set A. If $R^2 \subseteq R$, then R is transitive.

For a relation R to be transitive, that means that if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Proof: Note that $R^2 = R \circ R$. If $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition, $(a, c) \in R^2$. Because $R^2 \subseteq R$, this means that $(a, c) \in R$. In other words, R is transitive!