CSE 321 Autumn 2008 Final Exam Solutions

1. (10 points) For all integers m and n, prove that if $(m \cdot n)$ is even, then either m is even or n is even.

   Solution: Assume that $m \cdot n$ is even and m and n are both odd. Then $m = 2k + 1$ for some integer k and $n = 2j + 1$ for some integer j.

$$\begin{aligned} m \cdot n &= (2k+1)(2j+1) \\ &= 4kj + 2k + 2j + 1 \\ &= 2(kj + k + j) + 1 \end{aligned}$$

   Therefore $m \cdot n$ is odd. This is a contradiction. Thus either m is even or n is even.

2. (15 points) When it's not raining outside, Sam Slacker would rather go out and play. On such occasions, he is true to his name and does not do his homework 90% of the time. When it's raining, he has not much else to do, but he's still a slacker, so he only does his homework 40% of the time. Given that Sam lives in Seattle and it rains 80% of the time, what is the probability that it is raining given that Sam did his homework? Write your answer in lowest terms.

   Solution: Let H be the event that Sam does his homework and R be the event that it is raining. We know:

$$\begin{aligned} P(\neg H|\neg R) &= 0.9 \\ P(H|R) &= 0.4 \\ P(R) &= 0.8 \end{aligned}$$

   Since probabilities add up to 1, we also know:

$$\begin{aligned} P(H|\neg R) &= 0.1 \\ P(\neg H|R) &= 0.6 \\ P(\neg R) &= 0.2 \end{aligned}$$

$$\begin{aligned} P(R|H) &= \frac{P(H|R)P(R)}{P(H)} \\ &= \frac{P(H|R)P(R)}{P(H|R)P(R) + P(H|\neg R)P(\neg R)} \\ &= \frac{0.4 \cdot 0.8}{0.4 \cdot 0.8 + 0.1 \cdot 0.2)} \\ &= \frac{0.32}{0.32 + 0.02} = \frac{0.32}{0.34} \\ &= \frac{32}{34} = \frac{16}{17} \end{aligned}$$

3. (10 points) Show that the hypotheses $(p \wedge t) \rightarrow (r \vee s)$, $q \rightarrow (u \wedge t)$, $u \rightarrow p$, and $\neg s$ imply the conclusion $q \rightarrow r$.

   Solution: To prove an implication is true, we assume the hypothesis and derive the conclusion.

1.	q	Assumption
2.	$q \rightarrow (u \wedge t)$	Hypothesis
3.	$(u \wedge t)$	Modus Ponens from (1) and (2)
4.	u	Simplification from (3)
5.	$u \rightarrow p$	Hypothesis
6.	p	Modus Ponens from (4) and (5)
7.	t	Simplification from (3)
8.	$p \wedge t$	Conjunction from (6) and (7)
9.	$(p \wedge t) \rightarrow (r \vee s)$	Hypothesis
10.	$(r \vee s)$	Modus Ponens from (8) and (9)
11.	$\neg s$	Hypothesis
12.	r	Disjunctive Syllogism from (10) and (11)

4. (10 points)

   (a) (5 points) Let K_n be a complete graph (clique) with n vertices. For what values of n does K_n have an Euler circuit? Explain.

   Solution: A graph has an Euler circuit if and only if every vertex has even degree. The vertices in a complete graph have even degree when n is odd. Each edge is adjacent to the other $n-1$ vertices. $n-1$ is even when n is odd.

   (b) (5 points) How many graphs with exactly 3 edges can be made on a set of n vertices, where $n \geq 3$? Explain.

   Solution: There are $\binom{n}{2}$ possible edges on a graph with n vertices. Thus, there are $\binom{\binom{n}{2}}{3}$ graphs with exactly 3 of those edges.

5. (20 points)

   (a) (5 points) Suppose that R_1 and R_2 are equivalence relations on a set A. Prove or disprove that $R_1 \cup R_2$ is an equivalence relation.

   Solution: False. Let $A = \{1, 2, 3\}$. Define the following relations on A.

$$\begin{aligned} R_1 &= \{(1,1),(2,2),(3,3),(1,2),(2,1)\} \\ R_2 &= \{(1,1),(2,2),(3,3),(2,3),(3,2)\} \end{aligned}$$

   Their union is:

$$R_1 \cup R_2 = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}$$

   R_1 and R_2 are equivalence relations, but their union is not an equivalence relation, because it is not transitive: $(1,2)$ and $(2,3)$ are in the relation, but $(1,3)$ is not.

   (b) (5 points) Suppose that R_1 and R_2 are equivalence relations on a set A. Prove or disprove that $R_1 \cap R_2$ is an equivalence relation.

   Solution: True.

   reflexivity: Since R_1 is reflexive, then $\forall x \epsilon A (x, x) \epsilon R_1$. This is also the case for R_2, because R_2 is reflexive. Thus $(x, x) \epsilon R_1 \cap R_2 \forall x \epsilon A$.

   symmetry: If $(a, b) \epsilon R_1 \cap R_2$, then $(a, b) \epsilon R_1$ and $(a, b) \epsilon R_2$. Since both relations are symmetric $(b, a) \epsilon R_1$ and $(b, a) \epsilon R_2$. Consequently, $(b, a) \epsilon R_1 \cap R_2$.

   transitivity: If $(a, b) \epsilon R_1 \cap R_2$ and $(b, c) \epsilon R_1 \cap R_2$, then (a, b) and (b, c) are in both relations. Since both relations are also transitive, (a, c) is also in both relations. Thus $(a, c) \epsilon R_1 \cap R_2$.

   (c) (10 points) Define a relation R on a set A to be *reverse-transitive* if whenever $(a, c) \epsilon R$ and $(b, c) \epsilon R$, then $(a, b) \epsilon R$, for all $a, b, c \epsilon A$. Show that if R is reverse-transitive and reflexive, then it is an equivalence relation.

   Solution:

   reflexivity: Given.

   symmetry: Suppose $(a, b) \epsilon R$. Since R is reflexive, (b, b) is also in R. Since (b, b) and (a, b) are in R, (b, a) is in R by reverse-transitivity.

   transitivity: Suppose (a, b) and (b, c) are in R. (c, b) is in R by symmetry demonstrated above. (a, b) and (c, b) imply (a, c) by reverse-transitivity.

6. (10 points) A family gathering has twelve members. Each family member gives a gift to six other family members. Prove that some pair of individuals will have exchanged gifts. Hint: There are 66 such pairs of individuals.

   Solution: Let the 66 pairs of individuals be the pigeonholes. Since each of the twelve family members gives away 6 gifts, 72 gifts will be given from one person to another. Let that pair where a gift was given be a pigeon that goes to the pigeonhole associated with the same pair. By the pigeonhole principle, since there are more pigeons than pigeonholes, at least one pair will have at least 2 gifts given between the pair. Since each person gives only one gift maximum to any other person, the two gifts given must be from one person to the other and vice versa, i.e. a gift exchange between the individuals in a pair.

7. (20 points) On the SAT, each question has 5 multiple-choice answers. Your raw score is computed as follows: you get 1 point for every question answered correctly, 0 points for every unanswered question, and -0.25 points for every question answered incorrectly.

   (a) (10 points) A particular subject test has 48 questions. Assume you answer all the questions randomly, what is your expected raw score?

   Solution: Let X_i be the random variable representing the raw score of a question that is answered. X_i is 1 if you get the question right, and -0.25 if you get the question wrong. Since you are answering randomly, you have a $\frac{1}{5}$ chance of getting the question right and $\frac{4}{5}$ wrong. The expected value of getting any particular question right is:

$$\begin{aligned} E(X_i) &= p(X_i = r) \cdot r \\ &= \frac{1}{5} \cdot 1 + \frac{4}{5} \cdot (-\frac{1}{4}). \\ &= \frac{1}{5} - \frac{1}{5} \\ &= 0. \end{aligned}$$

   To get the expected raw score of the test, you multiply the number of questions by the expected score of each question. It doesn't matter how many questions there are, because each question has an expected score of 0. Neat, huh? The SAT was designed such that if you guess completely randomly, your expected score is 0!

   (b) (10 points) On another test of 48 questions, you are correctly able to eliminate two of the incorrect choices for each question in the first-half of the test. Of those questions, you then guess from the remaining three choices. For the other half of the test, you are correctly able to eliminate three of the possible answer choices and guess on the remaining two choices. What is your expected raw score?

   Solution: Let X_i be the random variable representing the raw score of a question on the first half of the test and X_j be the random variable representing the raw score of a question.

$$\begin{aligned} E(X_i) &= p(X_i = r) \cdot r = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot (-\frac{1}{4}) = \frac{1}{3} - \frac{2}{12} = \frac{1}{6} \\ E(X_j) &= p(X_j = r) \cdot r = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-\frac{1}{4}) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8} \end{aligned}$$

   Each half of the test has 24 questions. The expected raw score of the first half of the test is the number of questions times the expected raw score of each question: $24 \cdot \frac{1}{6} = 4$. The expected raw score of the second half of the test is $24 \cdot \frac{3}{8} = 9$. The total expected score is $4 + 9 = 13$.

8. (20 points)

   (a) (8 points) Consider a relation R over the set $S = \{a, b, c, d, e\}$. How many such relations are there that are reflexive? Explain. Try not to use more than 50 words (about 4 sentences).

   Solution: A relation is a subset of the Cartesian product $S \times S$. $S \times S$ has 25 tuples. However, R is reflexive, so that means all tuples of the form (x, x) where $x \epsilon S$ must be in R. There are 5 such tuples. That leaves 20 tuples to choose from when making a reflexive relation over S. Each tuple can either be in this reflexive relation or not. There are 2^{20} such relations.

   (b) (12 points) Suppose you chose a relation uniformly at random and the relation is reflexive. What is the probability that the size of the relation is at least 8 given that the relation is reflexive?

   Solution:

   Number of reflexive relations that have a cardinality less than 8:
   $\binom{20}{0} + \binom{20}{1} + \binom{20}{2}$
   Probability that a random reflexive relation has a cardinality less than 8:
   $\frac{\binom{20}{0}+\binom{20}{1}+\binom{20}{2}}{2^{20}}$
   Probability that a random reflexive relation has a cardinality at least 8:
   $1 - \frac{\binom{20}{0}+\binom{20}{1}+\binom{20}{2}}{2^{20}}$

9. (15 points) President-elect Obama is looking to fill his Cabinet. The Cabinet has 15 different positions. The list of candidates has 15 Republicans, 23 Democrats and 17 Independents. If selected for one position, a candidate may not be selected for another (in other words, candidates may hold at most one position). Assume that each candidate is fully qualified to fill any of the Cabinet positions.

   (a) (5 points) How many ways are there for Obama to choose his Cabinet?

   Solution: P(55,15)

   (b) (5 points) Considering only party affiliation, how many ways are there to form the Cabinet? For example, here are some ways to form the Cabinet:

   - 15 Republicans
   - 3 Republicans, 8 Democrats, and 4 Independents
   - 1 Republican, 1 Democrat, 13 Independents

   Solution: Since order doesn't matter and members of a party are indistinguishable from each other, this can be solved using stars and bars. There are 3 categories and 15 positions to be filled. $\binom{15+3-1}{2} = \binom{17}{2} = \binom{17}{15}$.

   (c) (5 points) If Obama wants at least 3 Democrats in his Cabinet, considering only party affiliation, how many ways are there to form the Cabinet?

   Solution: If 3 positions are reserved for Democrats, that leaves only 12 positions to vary the Cabinet. $\binom{12+3-1}{2} = \binom{14}{2} = \binom{14}{12}$.

10. Prove that $\sum_{j=2}^{n} \binom{j}{2} = \binom{n+1}{3}$ whenever n is an integer greater than 1,

(a) (10 points) using mathematical induction.

Solution:

Base case: $n = 2$:

$\sum_{j=2}^{2} \binom{j}{2} = \binom{2}{2} = 1$

$\binom{2+1}{3} = \binom{3}{3} = 1$

Inductive hypothesis: Assume $\sum_{j=2}^{k} \binom{j}{2} = \binom{k+1}{3}$

Inductive step:

$$\begin{aligned} \sum_{j=2}^{k+1} \binom{j}{2} &= \sum_{j=2}^{k} \binom{j}{2} + \binom{k+1}{2} \\ &= \binom{k+1}{3} + \binom{k+1}{2} \\ &= \binom{k+2}{3} \\ &= \binom{(k+1)+1}{3} \end{aligned}$$

The second to the last equality is due to Pascal's Identity, which could have also been derived mathematically.

Prove that $\binom{n+1}{3} = \sum_{j=2}^{n} \binom{j}{2}$ whenever n is an integer greater than 1,

(b) (10 points) using a combinatorial argument.

Solution: There are $n+1$ objects of which we want 3. These objects can be listed as $x_1, x_2, ..., x_{n+1}$. To pick 3 of them, we have to either include x_{n+1} as part of the 3 or we don't. If we do decide to pick it, then we need 2 more objects from the remaining n objects of which there are $\binom{n}{2}$ ways. If we do not decide to include x_{n+1}, we can make a similar decision about x_n. If we do include x_n, then there are $n-1$ objects of which to choose 2 more objects, or $\binom{n-1}{2}$ ways. There's only $n-1$ ways, because we already decided not to include x_{n+1}. So far, we have $\binom{n}{2} + \binom{n-1}{2}$ ways. We continue this reasoning down to x_3. When we get to this point, we have decided to not include all x_i for $i > 3$. While including x_3, there are two objects left and we need 2 of them, or $\binom{2}{2}$. The total count is $\binom{n}{2} + \binom{n-1}{2} + ... + \binom{2}{2} = \sum_{j=2}^{n} \binom{j}{2}$.