CSE 321 Discrete Structures

Winter 2008
Lecture 14
Recursive Definitions and Structural Induction

Announcements

- Readings
  - This week:
    - 6th edition: 4.3, 4.4, 5.1, 5.2
    - 5th edition: 3.4, 3.5, 4.1, 4.2
- Midterm:
  - Mean 67, Median 68

81+	4
71-80	10
61-70	16
51-60	7
0-50	1

Induction Example (revisited)

- Given a set S of n+1 positive integers, none exceeding 2n, show that S is divisible.
- Paul Beame's proof
  - Let $S \subseteq \{1, \ldots, 2n\}$ be non-divisible
  - Every element in S can be written as $m2^i$ where m is odd
  - We cannot have $m2^i$ and $m2^j$ both in S
  - Hence $|S| \leq n$

Highlights from Lecture 14

- Recursive Definitions
  - F(0) = 1; F(n+1) = 2*F(n)
  - $f_0 = 0$; $f_1 = 1$; $f_n = f_{n-1} + f_{n-2}$

Recursive Definitions of Sets

- Recursive definition
  - Basis step: $0 \in S$
  - Recursive step: if $x \in S$, then $x + 2 \in S$
  - Exclusion rule: Every element in S follows from basis steps and a finite number of recursive steps

Recursive definitions of sets

Basis: $6 \in S$; $15 \in S$;
Recursive: if $x, y \in S$, then $x + y \in S$;

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$;
Recursive:
if $[x, y, z] \in S$, α in R, then $[\alpha x, \alpha y, \alpha z] \in S$
if $[x_1, y_1, z_1], [x_2, y_2, z_2] \in S$
then $[x_1 + x_2, y_1 + y_2, z_1 + z_2]$

Powers of 3

Strings

- The set Σ^* of strings over the alphabet Σ is defined
  - Basis: $\lambda \in S$ (λ is the empty string)
  - Recursive: if $w \in \Sigma^*$, $x \in \Sigma$, then $wx \in \Sigma^*$

Families of strings over $\Sigma = \{a, b\}$

- L_1
  - $\lambda \in L_1$
  - $w \in L_1$ then $awb \in L_1$
- L_2
  - $\lambda \in L_2$
  - $w \in L_2$ then $aw \in L_2$
  - $w \in L_2$ then $wb \in L_2$

Function definitions

$Len(\lambda) = 0$;
$Len(wx) = 1 + Len(w)$; for $w \in \Sigma^*$, $x \in \Sigma$

$Concat(w, \lambda) = w$ for $w \in \Sigma^*$
$Concat(w_1,w_2x) = Concat(w_1,w_2)x$ for w_1, w_2 in Σ^*, $x \in \Sigma$

Well Formed Fomulae

- Basis Step
  - T, F, and s, where is a propositional variable are in WFF
- Recursive Step
  - If E and F are in WFF then $(\neg E)$, $(E \wedge F)$, $(E \vee F)$, $(E \rightarrow F)$ and $(E \leftrightarrow F)$ are in WFF

Tree definitions

- A single vertex r is a tree with root r.
- Let t_1, t_2, …, t_n be trees with roots r_1, r_2, …, r_n respectively, and let r be a vertex. A new tree with root r is formed by adding edges from r to r_1,…, r_n.

Extended Binary Trees

- The empty tree is a binary tree.
- Let r be a node, and T_1 and T_2 binary trees. A binary tree can be formed with T_1 as the left subtree and T_2 as the right subtree. If T_1 is non-empty, there is an edge from the root of T_1 to r. Similarly, if T_2 is non-empty, there is an edge from the root of T_2 to r.

Full binary trees

- The vertex r is a FBT.
- If r is a vertex, T_1 a FBT with root r_1 and T_2 a FBT with root r_2 then a FBT can be formed with root r and left subtree T_1 and right subtree T_2 with edges $r \rightarrow r_1$ and $r \rightarrow r_2$.

Simplifying notation

- $(\bullet, T_1, T_2)$, tree with left subtree T_1 and right subtree T_2
- ε is the empty tree
- Extended Binary Trees (EBT)
  - $\varepsilon \in$ EBT
  - if $T_1, T_2 \in$ EBT, then $(\bullet, T_1, T_2) \in$ EBT
- Full Binary Trees (FBT)
  - $\bullet \in$ FBT
  - if $T_1, T_2 \in$ FBT, then $(\bullet, T_1, T_2) \in$ FBT

Recursive Functions on Trees

- N(T) - number of vertices of T
- $N(\varepsilon) = 0$; $N(\bullet) = 1$
- $N(\bullet, T_1, T_2) = 1 + N(T_1) + N(T_2)$

- Ht(T) – height of T
- $Ht(\varepsilon) = 0$; $Ht(\bullet) = 1$
- $Ht(\bullet, T_1, T_2) = 1 + \max(Ht(T_1), Ht(T_2))$

NOTE: Height definition differs from the text
Base case $H(\bullet) = 0$ used in text

More tree definitions: Fully balanced binary trees

- ε is a FBBT.
- if T_1 and T_2 are FBBTs, with $Ht(T_1) = Ht(T_2)$, then $(\bullet, T_1, T_2)$ is a FBBT.

And more trees: Almost balanced trees

- ε is a ABT.
- if T_1 and T_2 are ABTs with $Ht(T_1) - 1 \leq Ht(T_2) \leq Ht(T_1)+1$ then $(\bullet, T_1, T_2)$ is a ABT.

Structural Induction

- Show P holds for all basis elements of S.
- Show that P holds for elements used to construct a new element of S, then P holds for the new elements.

Prove all elements of S are divisible by 3

- Basis: 6 $\in$ S; 15 $\in$ S;
- Recursive: if x, y $\in$ S, then x + y $\in$ S;

Prove that WFFs have the same number of left parentheses as right parentheses