CSE 321 Discrete Structures

Winter 2008
Lecture 15
Structural Induction

Announcements

- Readings
  - Today:
    - Structural Induction
      - 6th edition: 4.3, 5th edition: 3.4,
  - Friday:
    - Counting
      - 6th edition: 5.1, 5.2, 5th edition: 4.1, 4.2

Highlights from Lecture 14

- Recursive Definitions
  - Sets
    - $0 \in S$;
    - if $x \in S$ then $x+2 \in S$
  - Strings
    - $\lambda \in L$
    - $w \in L$, $x \in \{a, b\}$ then $wxx \in L$
  - Trees
    - $\varepsilon \in EBT$
    - if $T_1, T_2 \in EBT$, then $(\bullet, T_1, T_2) \in EBT$

Recursive Functions on Trees

- N(T) - number of vertices of T
- $N(\varepsilon) = 0$; $N(\bullet) = 1$
- $N(\bullet, T_1, T_2) = 1 + N(T_1) + N(T_2)$

- Ht(T) – height of T
- $Ht(\varepsilon) = 0$; $Ht(\bullet) = 1$
- $Ht(\bullet, T_1, T_2) = 1 + \max(Ht(T_1), Ht(T_2))$

NOTE: Height definition differs from the text
Base case $H(\bullet) = 0$ used in text

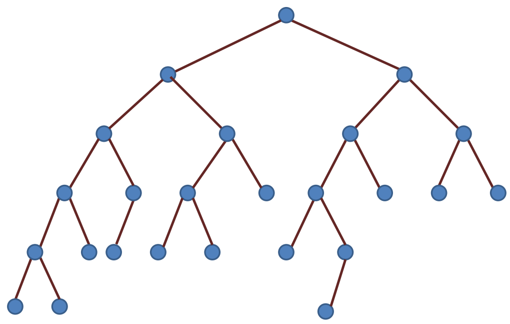
More tree definitions: Fully balanced binary trees

- ε is a FBBT.
- if T_1 and T_2 are FBBTs, with $Ht(T_1) = Ht(T_2)$, then $(\bullet, T_1, T_2)$ is a FBBT.

And more trees: Almost balanced trees

- ε is a ABT.
- if T_1 and T_2 are ABTs with $Ht(T_1) -1 \leq Ht(T_2) \leq Ht(T_1)+1$ then $(\bullet, T_1, T_2)$ is a ABT.

Is this Tree Almost Balanced?



Structural Induction

- Show P holds for all basis elements of S.
- Show that if P holds for elements used to construct a new element of S, then P holds for the new element.

Prove all elements of S are divisible by 3

- Basis: 21 $\in$ S; 24 $\in$ S;
- Recursive: if x, y $\in$ S, then x + y $\in$ S;

Prove that WFFs have the same number of left parentheses as right parentheses

Well Formed Fomulae

- Basis Step
  - T, F, and s, where is a propositional variable are in WFF
- Recursive Step
  - If E and F are in WFF then ($\neg$ E), (E$\wedge$ F), (E$\vee$ F), (E$\rightarrow$ F) and (E$\leftrightarrow$ F) are in WFF

Fully Balanced Binary Tree

- If T is a FBBT, then N(T) = $2^{Ht(T)} - 1$

Binary Trees

- If T is a binary tree, then $N(T) \le 2^{Ht(T)} - 1$

If $T = \varepsilon$:

If $T = (\bullet, T_1, T_2)$ $Ht(T_1) = x$, $Ht(T_2) = y$
$N(T_1) \le 2^x$, $N(T_2) \le 2^y$

$N(T) = N(T_1) + N(T_2) + 1$
$\le 2^x - 1 + 2^y - 1 + 1$
$\le 2^{Ht(T) -1} + 2^{Ht(T) - 1} - 1$
$\le 2^{Ht(T)} - 1$

Almost Balanced Binary Trees

Let $\alpha = (1 + sqrt(5))/2$
Prove $N(T) \ge \alpha^{Ht(T)} - 1$
Base case:
Recursive Case: $T = (\bullet, T_1, T_2)$
Let $Ht(T) = k + 1$
Suppose $Ht(T_1) \ge Ht(T_2)$
$Ht(T_1) = k$, $Ht(T_2) = k$ or $k-1$

Almost Balanced Binary Trees

$N(T) = N(T_1) + N(T_2) + 1$
$\ge \alpha^k - 1 + \alpha^{k-1} - 1 + 1$
$\ge \alpha^k + \alpha^{k-1} - 1$ $[\alpha^2 = \alpha + 1]$
$\ge \alpha^{k+1} - 1$