Discrete Structures

Counting

Chapter 5, Sections 5.1 - 5.4

Dieter Fox

Examples

$\diamondsuit$ Suppose that either a member of the mathematics faculty (37) or a mathematics student (113) is chosen as a representative to a university committee. How many different choices are there for the representative?

$\diamondsuit$ Chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

$\diamondsuit$ How many different bit strings are there of length seven?

$\diamondsuit$ How many subsets of a finite set S with $|S| = n$ are there?

Basic Principles

- **Product Rule:** Suppose that a procedure can be broken down into two tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.

- Alternative: The number of ways in which a sequence of events can occur is the product of the number of ways in which each individual event can occur.

- **Sum Rule:** If a first task can be done in n_1 ways and a second task in n_2 ways, and if these tasks cannot be done at the same time (i.e. they do not overlap), then there are $n_1 + n_2$ ways to do either task.

- Alternative: The number of ways in which a collection of mutually exclusive events can occur is the sum of the number of ways in which each event can occur.

More Examples

$\diamondsuit$ How many even numbers in 100-999 have no repeated digits?

$\diamondsuit$ How many functions are there from a set with m elements to a set with n elements?

$\diamondsuit$ How many one-to-one functions are there from a set with m elements to a set with n elements?

$\diamondsuit$ Each user on a computer system has a password of length six to eight characters. Each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Inclusion-Exclusion Principle

- ♢ Idea: Just like sum rule, but now the two tasks may be done at the same time (overlap): Add the number of ways to do each of the two tasks and then subtract the number of ways to do both tasks.

- ♢ Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Pigeonhole Principle

◇ **Pigeonhole Principle:** If $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

◇ **Generalized Pigeonhole Principle:** If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Permutations and Combinations

- A permutation of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an r-permutation.

- **Theorem:** The number of r-permutations of a set with n distinct elements, where n is a positive integer and r is an integer with $0 \leq r \leq n$, is $P(n,r) = \frac{n!}{(n-r)!}$.

- An r-combination of elements of a set is an unordered selection of r elements from the set.

- **Theorem:** The number of r-combinations of a set with n elements, where n is a positive integer and r is an integer with $0 \leq r \leq n$, is $C(n,r) = \frac{n!}{r!(n-r)!}$.

Examples

◇ In how many ways can ten adults and five children stand in a line so that no two children are next to each other?

◇ In how many ways can ten adults and five children stand in a circle so that no two children are next to each other?

◇ In how many ways can 20 students out of a class of 32 be chosen to attend class on a late Thursday afternoon (and take notes for the others) if

1. Paul refuses to go to class?
2. Michelle insists on going?
3. Jim and Michelle insist on going?
4. either Jim or Michelle (or both) go to class?
5. just one of Jim and Michelle attned?
6. Paul and Michelle refuse to attend class together?

Binomial Coefficients

◇ **Pascal's Identity:** Let n and k be positive integers with $n \geq k$. Then $C(n+1,k) = C(n,k-1) + C(n,k)$.

Alternative notation:
$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

◇ **Binomial Theorem:** Let x and y be variables, and let n be a positive integer. Then

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n\end{aligned}$$