What's on our platter today?

- Cliff's notes for equivalence of CFGs and PDAs
  - L = L(G) for some CFG G $\Rightarrow$ L = L(M) for some PDA M
  - L = L(M) for some PDA M $\Rightarrow$ L = L(G) for some CFG G
- Pumping Lemma (one last time)
  - Statement of Pumping Lemma for CFLs
  - Proof: On board and textbook
  - Application: Showing a given L is not a CFL

From CFGs to PDAs

- L is a CFL $\Rightarrow$ L = L(M) for some PDA M
- Proof Summary:
  - L is a CFL means L = L(G) for some CFG G = (V, Σ, R, S)
  - Construct PDA M = $(Q, \Sigma, \Gamma, \delta, q_0, \{q_{acc}\})$
    M has only 4 main states (plus a few more for pushing strings)
    $Q = \{q_0, q_1, q_2, q_{acc}\} \cup E$ where E are states used in 2 below
  - δ has 4 components:
  1. **Init. Stack**: $\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \$)\}$ and $\delta(q_1, \varepsilon, \varepsilon) = \{(q_2, S)\}$
  2. **Push & generate strings**: $\delta(q_2, \varepsilon, A) = \{(q_2, w)\}$ for $A \rightarrow w$ in R
  3. **Pop & match to input**: $\delta(q_2, a, a) = \{(q_2, \varepsilon)\}$
  4. **Accept if stack empty**: $\delta(q_2, \varepsilon, \$) = \{(q_{acc}, \varepsilon)\}$
- Can prove by induction: $w \in L$ iff $w \in L(M)$

From PDAs to CFGs

- L = L(M) for some PDA M $\Rightarrow$ L = L(G) for some CFG G
- Proof Summary: Simulate M's computation using a CFG
  - First, simplify M: 1. Only 1 accept state, 2. M empties stack before accepting, 3. Each transition either Push or Pop, not both or neither. Let $M = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{acc}\})$
  - Construct grammar $G = (V, \Sigma, R, S)$
  - Basic Idea: Define variables A_{pq} for simulating M
  - A_{pq} generates all strings w such that w takes M from state p with empty stack to state q with empty stack
  - Then, A_{q0qacc} generates all strings w accepted by M

From PDAs to CFGs (cont.)

- L = L(M) for some PDA M $\Rightarrow$ L = L(G) for some CFG G
- Proof (cont.)
  - Construct grammar $G = (V, \Sigma, R, S)$ where

    $V = \{A_{pq} \mid p, q \in Q\}$

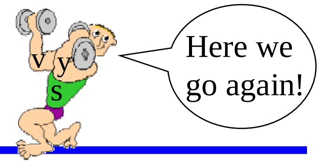
    $S = A_{q0qacc}$

    $R = \{A_{pq} \to aA_{rs}b \mid p \xrightarrow{a,\ \varepsilon \to c} r \overset{A_{rs}}{\dashrightarrow} s \xrightarrow{b,\ c \to \varepsilon} q\}$

    $\cup \{A_{pq} \to A_{pr} A_{rq} \mid p, q, r \in Q\}$

    $\cup \{A_{qq} \to \varepsilon \mid q \in Q\}$
- See text for proof by induction: $w \in L(M)$ iff $w \in L(G)$
- Try to get G from M where $L(M) = \{0^n1^n \mid n \geq 1\}$

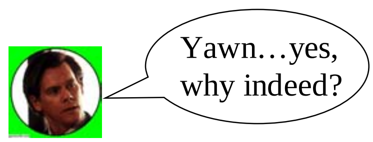
Pumping Lemma for CFLs



- Intuition: If L is CF, then some CFG G produces strings in L
  - If some string in L is very long, it will have a very tall parse tree
  - If a parse tree is taller than the number of distinct variables in G, then *some variable* A *repeats* ⇒ A will have at least two sub-trees
  - We can pump up the original string by replacing A's smaller sub-tree with larger, and pump down by replacing larger with smaller
- Pumping Lemma for CFLs in all its glory:
  - If L is a CFL, then there is a number p (the "pumping length") such that for all strings *s* in L such that $|s| \geq$ p, there exist *u*, *v*, *x*, *y*, and *z* such that *s* = *uvxyz* and:

1. $uv^ixy^iz \in$ L for all $i \geq 0$, and
2. $|vy| \geq 1$, and
3. $|vxy| \leq$ p.

Why is the PL useful?



- Can use the pumping lemma to show a language L is *not context-free*
  - <u>5 steps for a proof by contradiction</u>:
  1. Assume L is a CFL.
  2. Let p be the pumping length for L given by the pumping lemma for CFLs.
  3. Choose cleverly an *s* in L of length at least p, such that
  4. For *all possible ways* of decomposing *s* into *uvxyz*, where $|vy| \geq 1$ and $|vxy| \leq$ p,
  5. Choose an $i \geq 0$ such that uv^ixy^iz is not in L.
- Examples: Show the following are not CFLs
  - L = $\{0^n1^n0^n \mid n \geq 0\}$ and L = $\{0^n \mid$ n is a prime number$\}$

Using the Pumping Lemma

Oh boy…
Jolly good

- Show $L = \{0^n \mid n \text{ is a prime number}\}$ is not a CFL
  1. Assume L is a CFL.
  2. Let p be the pumping length for L given by the pumping lemma for CFLs.
  3. Let $s = 0^n$ where n is a prime $\geq p$
  4. Consider *all possible ways* of decomposing *s* into *uvxyz*, where $|vy| \geq 1$ and $|vxy| \leq p$.
     Then, $vy = 0^r$ and $uxz = 0^q$ where $r + q = n$ and $r \geq 1$
  5. We need an $i \geq 0$ such that $uv^ixy^iz = 0^{ir+q}$ is not in L.
     ($i = 0$ won't work because q could be prime: e.g. 2 + 17 = 19)
     Choose $i = (q + 2 + 2r)$. Then, $ir + q = qr + 2r + 2r^2 + q = q(r+1) + 2r(r+1) = (q+2r)(r+1)$ = not prime (since $r \geq 1$).

  So, 0^{ir+q} is not in L $\Rightarrow$ contradicts pumping lemma. L is not a CFL.

Two surprising results about CFLs

- CFLs are <u>not closed</u> under intersection
  - **Proof**: $L_1 = \{0^n1^n0^m \mid n, m \geq 0\}$ and $L_2 = \{0^m1^n0^n \mid n, m \geq 0\}$ are both CFLs but $L_1 \cap L_2 = \{0^n1^n0^n \mid n \geq 0\}$ is not a CFL.
- CFLs are <u>not closed</u> under complementation
  - **Proof by contradiction**:
    Suppose CFLs are closed under complement.

    Then, for L_1, L_2 above, $\overline{\overline{L_1} \cup \overline{L_2}}$ must be a CFL (since CFLs are closed under $\cup$ -- see homework #5, problem 1).

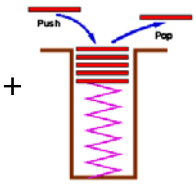
    But, $\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2$ (by de Morgan's law).

    $L_1 \cap L_2 = \{0^n1^n0^n \mid n \geq 0\}$ is not a CFL $\Rightarrow$ contradiction.

    Therefore CFLs are not closed under complementation.

Can we make PDAs more powerful?

- PDA = NFA +



- What if we allow arbitrary reads/writes to the stack instead of only push and pop?

Enter…Turing Machines (Next Class)

- Homework #5 due on Friday!