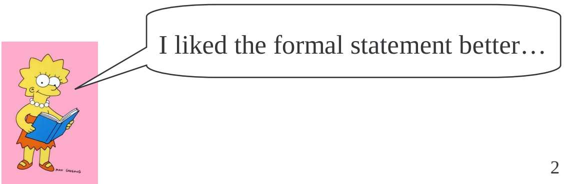
Pumping Lemma Recap

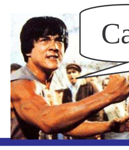
- **Formal Statement of the Pumping Lemma**: If L is regular, then ∃ p such that ∀ *s* in L with |*s*| ≥ p, ∃ *x*, *y*, *z* with *s* = *xyz* and:
  1. $xy^iz \in L \; \forall \; i \geq 0$, and
  2. $|y| \geq 1$, and
  3. $|xy| \leq p$.
- Proof on board last time…(see also page 79 in textbook)
- Proved in 1961 by Bar-Hillel, Peries and Shamir

Pumping Lemma in Plain English

- Let L be a regular language and let p = "pumping length" = no. of states of a DFA accepting L
- Then, any string *s* in L of length ≥ p can be expressed as *s* = *xyz* where:
  - *y* is not empty (*y* is the cycle)
  - |*xy*| ≤ p (cycle occurs within p state transitions), and
  - any "pumped" string xy^iz is also in L for all $i \geq 0$ (go through the cycle 0 or more times)



Using The Pumping Lemma



- In-Class Examples: Using the Pumping Lemma to show a language L is *not regular*
  - <u>5 steps for a proof by contradiction</u>:
  1. Assume L is regular. Then, L satisfies the P. Lemma.
  2. Let p be the pumping length given by the P. Lemma.
  3. Choose cleverly an s in L of length at least p, such that:
  4. For *all ways* of decomposing s into xyz, where $|xy| \leq p$ and y is not empty,
  5. There is an $i \geq 0$ such that xy^iz is not in L.

Proving non-regularity as a Two-Person game

- An alternate view: Think of it as a *game between you and an opponent (JC)*:
  1. **You**: Assume L is regular
  2. **JC**: Chooses some value p
  3. **You**: Choose cleverly an s in L of length $\geq$ p
  4. **JC**: Breaks s into some xyz, where $|xy| \leq p$ and y is not empty,
  5. **You**: Need to choose an $i \geq 0$ such that xy^iz is not in L (in order to win (the prize of non-regularity)!)

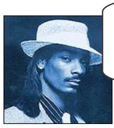
  (<u>Note</u>: Your i should work for all xyz that JC chooses, given your s)

Proving Non-Regularity using the Pumping Lemma

- Examples: Show the following are not regular
  - $L_1 = \{0^n1^n \mid n \geq 0\}$ over the alphabet {0, 1}
  - $L_2 = \{w \mid w$ contains equal number of 0s and 1s$\}$ over the alphabet {0, 1}
  - $L_3 = \{0^n1^m \mid n > m\}$ over the alphabet {0, 1}
  - ADD = {x=y+z | x, y, z are binary numbers and x is the sum of y and z} over the alphabet {0, 1, =, +}
  - SQUARES = $\{0^m \mid m = n^2$ for some $n \geq 0\}$ over alphabet {0} (see textbook for the proof)

Da Pumpin' Lemma

(Orig. lyrics: Harry Mairson)



Hear it on my new album:
Dig dat funky DFA

Any regular language *L* has a magic numba *p*
And any long-enuff word *s* in *L* has da followin' propa'ty:
Amongst its first *p* symbols is a segment you can find
Whoz repetition or omission leaves *s* amongst its kind.

So if ya find a language *L* which fails dis acid test,
And some long word ya pump becomes distinct from all da rest,
By contradiction you have shown dat language *L* is not
A regular homie, resilient to the damage you've caused.

But if, upon the other hand, *s* stays within its *L*,
Then either *L* is regulah, or else you chose not well.
For *s* is *xyz*, where *y* cannot be empty,
And *y* must come before da *p+1**th* symbol is read.

If $\{0^n1^n \mid n \geq 0\}$ is not Regular, what is it?

Irregular??

Enter…the world of Grammars
(after the Midterm)

CSE 322: Midterm Review

- **Basic Concepts** (Chapter 0)
  - Sets
    - Notation and Definitions
      - $A = \{x \mid \text{rule about } x\}$, $x \in A$, $A \subseteq B$, $A = B$
      - $\exists$ ("there exists"), $\forall$ ("for all")
    - Finite and Infinite Sets
      - Set of natural numbers N, integers Z, reals R etc.
      - Empty set $\emptyset$
    - Set operations: Know the definitions for proofs
      - Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
      - Intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
      - Complement $\overline{A} = \{x \mid x \notin A\}$

Basic Concepts (cont.)

- Set operations (cont.)
  - Power set of A = Pow(A) or 2^A = set of all subsets of A
    - E.g. A = {0,1} $\Rightarrow$ 2^A = {∅, {0}, {1}, {0,1}}
  - Cartesian Product $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$
- Functions:
  - f: Domain $\rightarrow$ Range
    - Add(x,y) = x + y $\Rightarrow$ Add: $Z \times Z \rightarrow Z$
  - Definitions of 1-1 and onto (bijection if both)

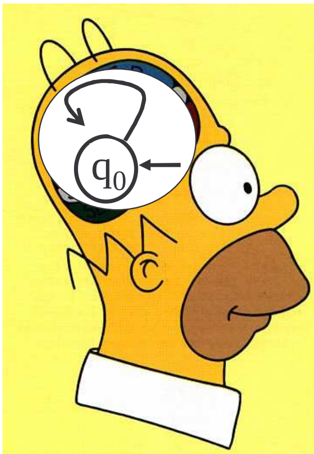
Strings

- Alphabet Σ = finite set of symbols, e.g. $\Sigma = \{0,1\}$
- String w = finite sequence of symbols $\in \Sigma$
  - $w = w_1 w_2 \ldots w_n$
- String properties: Know the definitions
  - Length of w = |w| (|w| = n if $w = w_1 w_2 \ldots w_n$)
  - Empty string = ε (length of ε = 0)
  - Substring of w
  - Reverse of w = $w^R = w_n w_{n-1} \ldots w_1$
  - Concatenation of strings x and y (append y to x)
  - y^k = concatenate y to itself to get string of *k* y's
  - Lexicographical order = order based on length and dictionary order within equal length

Languages and Proof Techniques

- Language L = set of strings over an alphabet (i.e. $L \subseteq \Sigma^*$)
  - E.g. $L = \{0^n1^n \mid n \geq 0\}$ over $\Sigma = \{0,1\}$
  - E.g. L = {p | p is a syntactically correct C++ program} over Σ = ASCII characters
- Proof Techniques: Look at lecture slides, handouts, and notes
  1. Proof by counterexample
  2. Proof by contradiction
  3. Proof of set equalities (A = B)
  4. Proof of "iff" ($X \Leftrightarrow Y$) statements (prove both $X \Rightarrow Y$ and $X \Leftarrow Y$)
  5. Proof by construction
  6. Proof by induction
  7. Pigeonhole principle
  8. Dovetailing to prove a set is countably infinite E.g. Z or $N \times N$
  9. Diagonalization to prove a set is uncountable E.g. 2^N or Reals

Chapter 1 Review: Languages and Machines



Languages and Machines (Chapter 1)

- Language = set of strings over an alphabet
  - Empty language = language with no strings = $\varnothing$
  - Language containing only empty string = $\{\varepsilon\}$
- DFAs
  - Formal definition $M = (Q, \Sigma, \delta, q_0, F)$
  - Set of states Q, alphabet Σ, start state q_0, accept ("final") states F, transition function $\delta: Q \times \Sigma \to Q$
  - M recognizes language $L(M) = \{w \mid M \text{ accepts } w\}$
  - In class examples:
    E.g. DFA for $L(M) = \{w \mid w \text{ ends in } 0\}$
    E.g. DFA for $L(M) = \{w \mid w \text{ does not contain } 00\}$
    E.g. DFA for $L(M) = \{w \mid w \text{ contains an even \# of 0's}\}$

  Try: DFA for $L(M) = \{w \mid w \text{ contains an even \# of 0's and an odd number of 1's}\}$

Languages and Machines (cont.)

- Regular Language = language recognized by a DFA
- Regular operations: Union $\cup$, Concatenation $\circ$ and star *
  - Know the definitions of $A \cup B$, $A \circ B$ and A^*
  - $\Sigma = \{0,1\} \Rightarrow \Sigma^* = \{\varepsilon, 0, 1, 00, 01, \ldots\}$
- Regular languages are closed under the regular operations
  - Means: If A and B are regular languages, we can show $A \cup B$, $A \circ B$ and A^* (and also B^*) are regular languages
  - Cartesian product construction for showing $A \cup B$ is regular by simulating DFAs for A and B in parallel
- Other related operations: $A \cap B$ and complement $\overline{A}$
  - Are regular languages closed under these operations?

NFAs, Regular expressions, and GNFAs

- NFAs vs DFAs
  - DFA: δ(state,symbol) = next state
  - NFA: δ(state,symbol or ε) = set of next states
    - Features: Missing outgoing edges for one or more symbols, multiple outgoing edges for same symbol, ε-edges
  - Definition of: NFA N accepts a string $w \in \Sigma^*$
  - Definition of: NFA N recognizes a language $L(N) \subseteq \Sigma^*$
  - E.g. NFA for $L = \{w \mid w = x1a, x \in \Sigma^* \text{ and } a \in \Sigma\}$
- Regular expressions: Base cases ε, $\varnothing$, $a \in \Sigma$, and $R1 \cup R2$, $R1 \circ R2$ or $R1^*$
- GNFAs = NFAs with edges labeled by regular expressions
  - Used for converting NFAs/DFAs to regular expressions

Main Results and Proofs

- L is a Regular Language iff
  - L is recognized by a DFA iff
  - L is recognized by an NFA iff
  - L is recognized by a GNFA iff
  - L is described by a Regular Expression
- Proofs:
  - NFA$\rightarrow$DFA: subset construction (1 DFA state=subset of NFA states)
  - DFA$\rightarrow$GNFA$\rightarrow$Reg Exp: Repeat two steps:
    1. Collapse two parallel edges to one edge labeled $(a \cup b)$, and
    2. Replace edges through a state with a loop with one edge labeled (ab^*c)
  - Reg Exp$\rightarrow$NFA: combine NFAs for base cases with ε-transitions

Other Results

- Using NFAs to show that Regular Languages are closed under:
  - Regular operations $\cup$, $\circ$ and *
- Are Regular Languages closed under:
  - intersection?
  - complement (Exercise 1.10)?
- Are there other operations that regular languages are closed under?

What about the reversal operation?

What about the idon’tcare operation?

What about the subset operation?

Other Results

- Are Regular languages closed under:
  - reversal?
  - subset ($\subseteq$) ?
  - superset ($\supseteq$) ?
  - Prefix?

    Prefix(L) = $\{w \mid w \in \Sigma^* \text{ and } wx \in L \text{ for some } x \in \Sigma^*\}$
  - NoExtend?

    NoExtend(L) = $\{ w \mid w \in L \text{ but } wx \notin L \text{ for all } x \in \Sigma^*-\{\varepsilon\}\}$
    (see also Problem 1.32 in the text)

Pumping Lemma

- *Pumping lemma in plain English (sort of):* If L is regular, then there is a p (= number of states of a DFA accepting L) such that any string *s* in L of length $\geq$ p can be expressed as $s = xyz$ where *y* is not null (*y* is the loop in the DFA), $|xy| \leq p$ (loop occurs within p state transitions), and any "pumped" string xy^iz is in L for all $i \geq 0$ (go through the loop 0 or more times).
- *Pumping lemma in plain Logic:*
  L regular $\Rightarrow \exists p$ s.t. $(\forall s \in L$ s.t. $|s| \geq p$ $(\exists x,y,z \in \Sigma^*$ s.t. $(s = xyz)$ and $(|y| \geq 1)$ and $(|xy| \leq p)$ and $(\forall i \geq 0, xy^iz \in L)))$
- Is the other direction $\Leftarrow$ also true?

  No! See Problem 1.37 for a counterexample

Proving Non-Regularity using the Pumping Lemma

- Proof by contradiction to show L is not regular
  1. Assume L is regular. Then L must satisfy the P. Lemma.
  2. Let p be the "pumping length"
  3. Choose a long enough string $s \in L$ such that $|s| \geq p$
  4. Let x,y,z be strings such that $s = xyz$, $|y| \geq 1$, and $|xy| \leq p$
  5. Pick an $i \geq 0$ such that $xy^iz \notin L$ (for *all possible* x,y,z as in 4)

  This contradicts the P. lemma. Therefore, L is not regular
- Examples: $\{0^n1^n | n \geq 0\}$, $\{ww | w \in \Sigma^*\}$, $\{0^m | m=n^2\}$, ADD = $\{x=y+z \mid$ x, y, z are binary numbers and x is sum of y and z$\}$
- Can sometimes also use closure under $\cap$ (and/or complement)
  - E.g. If $L \cap B = L_1$, and B is regular while L_1 is not regular, then L is also not regular (if L was regular, L_1 would be regular)

Some Applications of Regular Languages

- Pattern matching and searching:
  - E.g. In Unix:
    - `ls *.c`
    - `cp /myfriends/games/*.* /mydir/`
    - `grep 'Spock' *trek.txt`
- Compilers:
  - `id ::= letter (letter | digit)*`
  - `int ::= digit digit*`
  - `float ::= d d*.d*(ε|E d d*)`
  - The symbol | stands for "or" (= union)

Good luck on the midterm on Wednesday!

- You can bring one 8 1/2" x 11" review sheet (double-sided ok)
- The questions sheet will have space for answers. We will also bring extra blank sheets for those of you who don't believe in brevity.

Don't sweat it! 

- Go through the homeworks, lecture slides, and examples in the text (Chapters 0 and 1 only)
- Do the practice midterm on the website and avoid being surprised!