Notes for Friday, April 30th

We have previously looked at these examples: $L_1 = \{0^n1^n | n \geq 0\}$ and $L_2 = \{ww | w \in \{0,1\}^*\}$. Now we will look at more examples.

Example: show that $L_P = \{0^n | n \text{ is prime}\}$ is not regular. (Remark: this shows that a language with a single symbol in its alphabet doesn't have to be simple)

Proof (by contradiction):

1. Assume L_P is regular
2. There exists a p as in the Pumping Lemma
3. Choose $s \in L_P$ such that $|s| \geq p$, so $s = 0^k$ with k prime, $k \geq p+2$ (we use $p+2$ rather than p to show a result later. We may always find such a k because there are infinitely many primes.)
4. For any x, y, z such that $s = xyz$, $|y| \geq 1$ and $|xy| \leq p$.
5. We need to choose i such that $xy^iz \notin L_P$. What does this mean? We want $xy^iz = 0^m$ such that m is not prime, or that $m = n_1 \times n_2$ with $n_1, n_2 \geq 2$. Hence, we want $|xy^iz| = |xz| + i|y| = n_1n_2$ for such n_1, n_2.

   Choose $i = |xz|$. Then, $|zy^iz| = |xz| + |xz||y| = |xz|(1 + |y|)$. We have $n_1 = |xz|$ and $n_2 = 1 + |y|$. Notice that $|y| \geq 1$, so $n_2 \geq 2$. Also, $|zy| \leq p$ and $|xyz| \geq p+2$ by the choice of s and the assertions of the Pumping Lemma, so $|z| \geq 2$, or $|xz| = n_1 \geq 2$, as desired. We have shown that for our choice $i = |xz|$, $|xy^iz|$ is not prime and thus $s \notin L_P$, which is a contradiction.

There are some more useful tricks.

Example: Show that $L_3 = \{w | w \in \{0,1\}^*$ and w contains an equal number of 0s and 1s$\}$ is not regular.

We could prove it like we proved L_1 is not regular, using the same string. *Or,* we could notice that $L_3 \cap 0^*1^* = L_1$. If L_3 were regular, by the closure properties of regular languages, the intersection of the two regular languages L_3 and 0^*1^* would be regular, but this intersection (L_1) is not regular, which is a contradiction. Hence, L_3 is not regular.

Example: (Language was also called "Distinct" in lecture): Show that $L = \{w | w = x_1\#x_2\# \cdots \#x_k, k \geq 0, x_i \in 1^*, x_i \neq x_j \text{ for } i \neq j\}$. (A collection of strings that have an unequal number of 1s)

Assume L is regular. Then, $\bar{L} \cup 1^*\#1^* = \{1^n\#1^n\}$, but this result is not regular (proof is similar to that for language L_1), so by the closure properties of languages this is a contradiction. Hence, L is not regular.

In particular, we may use the that regular languages are closed under complement, intersection, union, concatenation, and star.