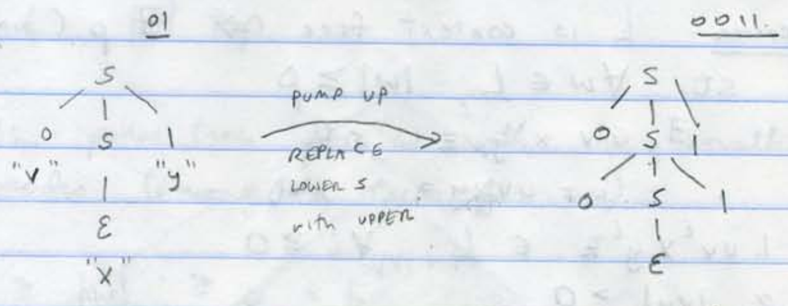
5/19/2010

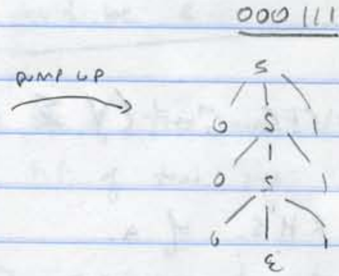
Pumping Lemma for CFL's

$L = \{ 0^n 1^n \mid n \geq 0 \}$

$S \rightarrow 0S1 \mid \varepsilon$

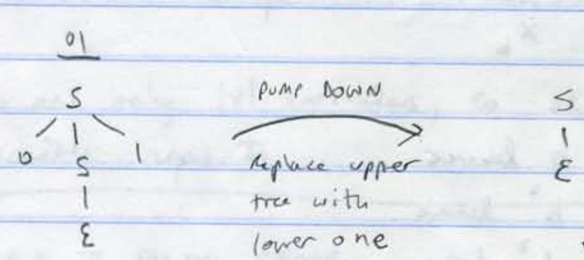


$vxy \in L$ $\qquad$ $v^2xy^2 \in L$



$v^3xy^3 \in L$

$v^ixy^i \in L, \ \forall i \geq 1$ $\qquad$ What about $i = 0$?



$v^0xy^0 \in L$

Yes, $i = 0$ also works.

Proof

Pumping Lemma: L is context free $\Rightarrow \exists\, p$ (pumping length)

s.t. $\forall w \in L,\ |w| \geq p$

$\exists\ u, v, x, y, z$ s.t.

$w = uvxyz$ AND

1. $uv^i x y^i z \in L \quad \forall i \geq 0$
2. $|vy| > 0$
3. $|vxy| \leq p$

Pf.

L is C.F. $\Rightarrow L = L(G)$ for CFG $G = (V, \Sigma, R, S)$

① Let b = max # of symbols on R.H.S. of a rule in R:

$A \to X_1 X_2 \cdots X_b, \quad X_i \in V$

# of branches

Ht 1 $\Rightarrow \leq b$ leaves

Ht 2 $\Rightarrow \leq b^2$ leaves

Ht 3 $\Rightarrow \leq b^3$ leaves

⋮

$h \Rightarrow \leq b^h$ leaves (can be proved by induction)

Let $\rho = b^{|V|+1}$

Let $w \in L$, $|w| \geq \rho$

Let T be a parse tree for w with the smallest # of nodes (among all trees for w)

$\underbrace{\# \text{ Leaves} \geq |w|}_{\text{because some leaves could be } \varepsilon} \geq \rho = b^{|V|+1}$

THE REASONING

If ht of tree was $|V|+1 \Rightarrow \leq b^{|V|+1}$ leaves
If ht of tree was $|V| \Rightarrow \leq b^{|V|}$ leaves

$\Rightarrow$ minimum height of tree is $|V|+1 \Rightarrow$
there is at least one path of length $|V|+1 \Rightarrow$
this path has $|V|+2$ nodes,
the last node is a terminal $\Rightarrow$
the path contains at least $|V|+1$ variables

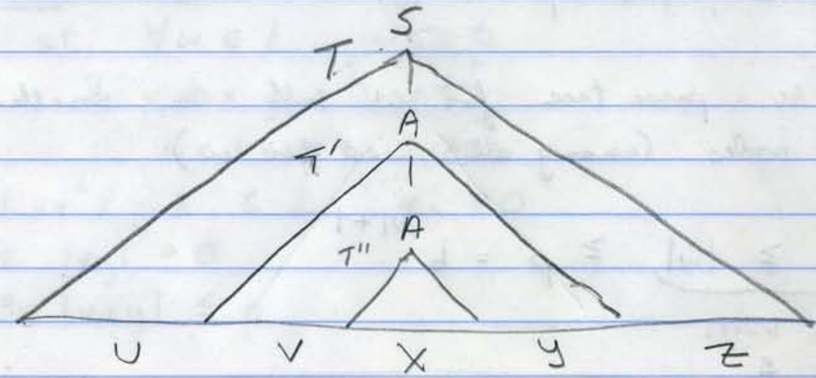
There are only $|V|$ variables, so at least one variable repeats.

BACK TO FORMAL PROOF...

T's ht $\geq |V|+1$
$\Rightarrow$ T has a path of $|V|+2$ nodes, Last is a terminal
$|V|+1$ nodes are variables.
There are only $|V|$ variables
By pigeonhole principle, some var A repeats

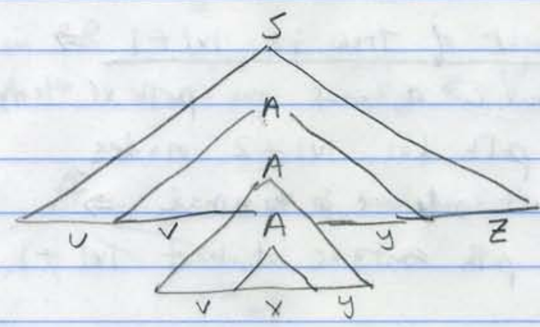
Choose A in lowermost $|V|+1$ vars on path

leaves $w =$



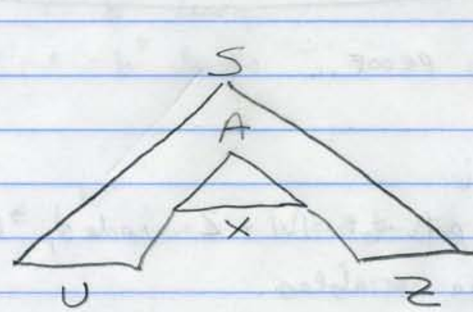
<u>Pump up</u>: Replace lower A w/ upper

$uv^2xy^2z \in L$



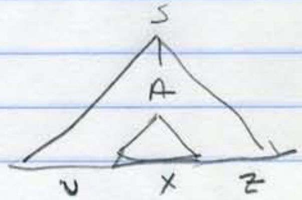
<u>Pump down</u>: Replace upper tree with lower

$uv^0xy^0z \in L$



② $|vy| > 0$

Suppose $|vy| = 0 \Rightarrow v = \varepsilon, y = \varepsilon$



$w = uxz$

$\Rightarrow \exists$ a smaller tree for w than T

$\Rightarrow$ contradicts T is smallest tree for w

③ $|vxy| \leq p$

A is in lowermost $|V|+1$ vars on path.

A is at height $\leq |V|+1$ from bottom

T' has $\leq \underset{\underset{p}{\parallel}}{b}^{|V|+1}$ leaves