Notes for Wednesday, June 2nd

Recall: $A_{TM} = \{\langle M, w\rangle | M$ is a TM and M accepts $w\}$. A_{TM} s Turing-recognizable (via Universal TM) but not decidable (via diagonalization technique).

Now we ask the question: is there a language that is not even Turing-recognizable.

Suppose $\overline{A_{TM}}$ is also Turing-recognizable.

Theorem: L is decidable iff L and $\bar{L}$ are Turing recognizable

Proof:

($\Rightarrow$) All decidable languages are Turing-recognizable, so L is Turing-recognizable. If L is decidable, that automatically implies that L is Turing-recognizable.If L is decidable, $\bar{L}$ is also decidable (decidable languages are closed under complement), so $\bar{L}$ is also Turing-recognizable.

($\Leftarrow$) If L and $\bar{L}$ are Turing-recognizable, then there exist M_1 and M_2 such that $L(M_1) = L$ and $L(M_2) = \bar{L}$. We can construct a decider TM for L:

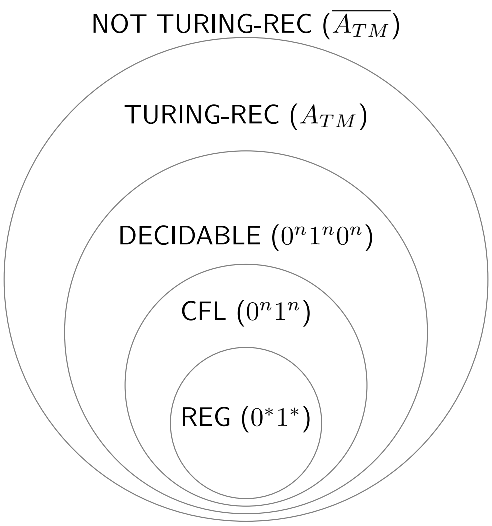
"on input w:
run M_1 and M_2 on w by alternating one step at a time
If M_1 accepts, M accepts If M_2 accepts, M rejects"

This way, M is guaranteed to half on all inputs (because the string is either in L or $\bar{L}$, and because M_1 and M_2 are run in parallel, it doesn't matter if one of them goes into an infinite loop). Thus, L is decidable.

Corollary: $\overline{A_{TM}}$ is not Turing-recognizable.

(If it were, A_{TM} itself would be decidable by the theorem, which is a contradiction)

This is the Chomsky hierarchy of problems:



$\overline{A_{TM}}$ is undecidable; are there more such problems?

Suppose you want to show that B is undecidable, and you know that A is undecidable. If you can use B to solve A (B is a decider for A), then A is decidable and this is a contradiction.

In this way, you can reduce an undecidable problem A to another problem B. If B is decidable, then there is a contradiction.

The notion is to use the new problem B to solve the original problem A

Notation: A is reducible to B if you can use B to solve A. We write $A \leq B$.

Suppose $B \leq C$, and $C \leq D$. Then we can write $A \leq B \leq C \leq D$.

Let $E_{TM} = \{\langle M\rangle | M$ is a TM and $L(M) = \emptyset\}$.

Theorem: $A_{TM} \leq E_{TM}$ (this E_{TM} is undecidable, by reduction)

Proof: Assume E_{TM} is decidable. Then, there exists a decider TM M_E such that $L(M_E) = E_{TM}$.

Construct a decided for A_{TM} as follows:

"on input $\langle M, w\rangle$,

1. Build TM M_1 on input x:

   (a) If $x \neq w$, reject
   (b) If $x = w$, then simulate M on w, accept if M accepts

   (then $L(M_1) = \{\{w\}$ if M accepts $w, \emptyset$ otherwise$\}$)

2. Feed M_1 to M_E

3. Accept $\langle M, w \rangle$ if M_E rejects $\langle M_1 \rangle$; Reject $\langle M, w \rangle$ if M_E accepts $\langle M_2 \rangle$."

This is a contradiction, so E_{TM} is undecidable.