Solving Problems with Turing Machines

- We know $L = \{0^n1^n0^n \mid n \geq 0\}$ is not a CFL (pumping lemma)
- Can we show L is decidable?
  - Construct a decider M such that L(M) = L
  - A <u>decider</u> is a TM that always halts (in q_{acc} or q_{rej}) and is *guaranteed not to go into an infinite loop for any input*

Input: 000001111100000

Idea: Mark off matching 0s, 1s, and 0s with Xs (left end marked with blank)

000001111100000
_0000111100000
_0000X111100000
_0000X1111X0000
_X000X1111X0000
….

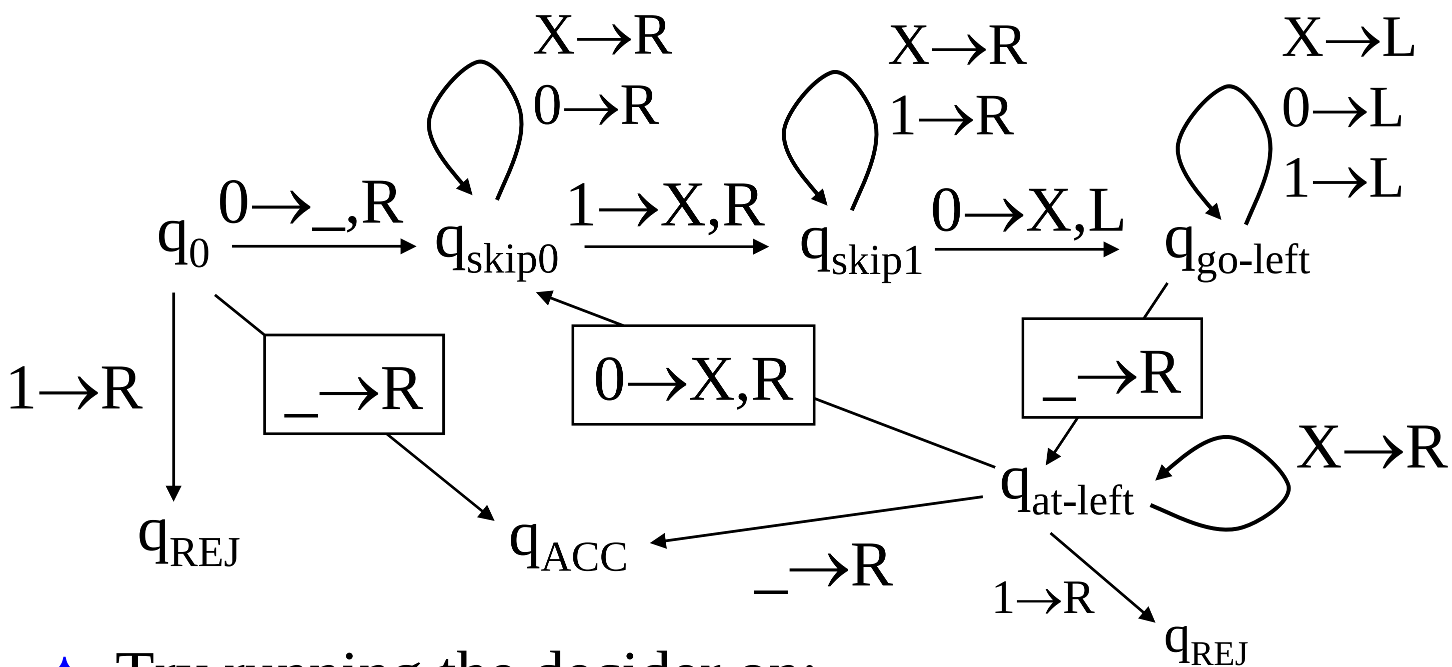
Idea for a Decider for $\{0^n1^n0^n \mid n \geq 0\}$

- **General Idea**: Match each 0 with a 1 and a 0 following the 1.
- Implementation Level Description of a Decider for L:

On input w:

1. If first symbol = blank, ACCEPT
2. If first symbol = 1, REJECT
3. If first symbol = 0, Write a blank to mark left end of tape
   a. If current symbol is 0 or X, skip until it is 1. REJECT if blank.
   b. Write X over 1. Skip 1's/X's until you see 0. REJECT if blank.
   c. Write X over 0. Move back to left end of tape.
4. At left end: Skip X's until:
   a. You see 0: Write X over 0 and GOTO 3a
   b. You see 1: REJECT
   c. You see a blank space: ACCEPT

State Diagram

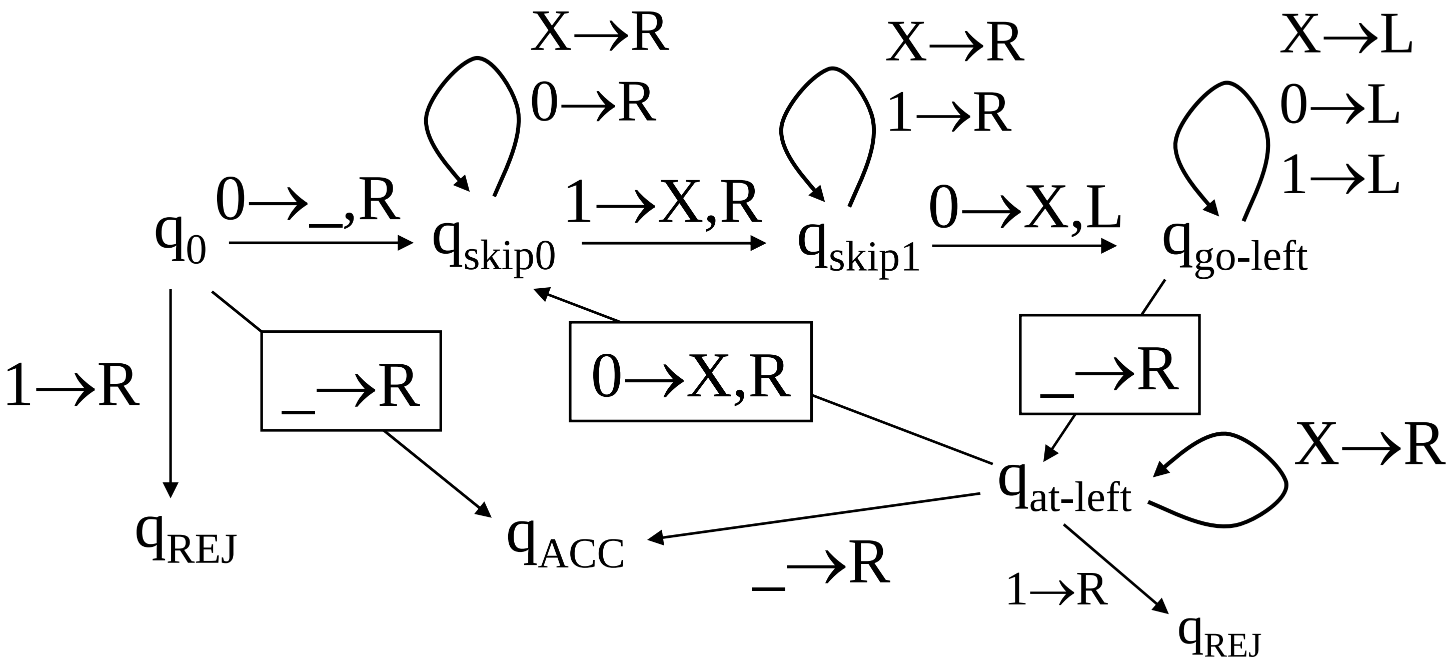


- Try running the decider on:
  - 010, 001100, … → ACCEPT
  - 0, 000, 0100, … → REJECT
  - What about 010010?

Note: Some transitions to q_{REJ} (e.g., from q_{skip0}) are not shown to avoid clutter

Houston, we have a problem…with our Turing machine.

What's the problem?



- The decider accepts incorrect strings:
  - 010010, 010001100 → ACCEPT!!!
  - Accepts $(0^n1^n0^n)^*$

Need to fix it…How??

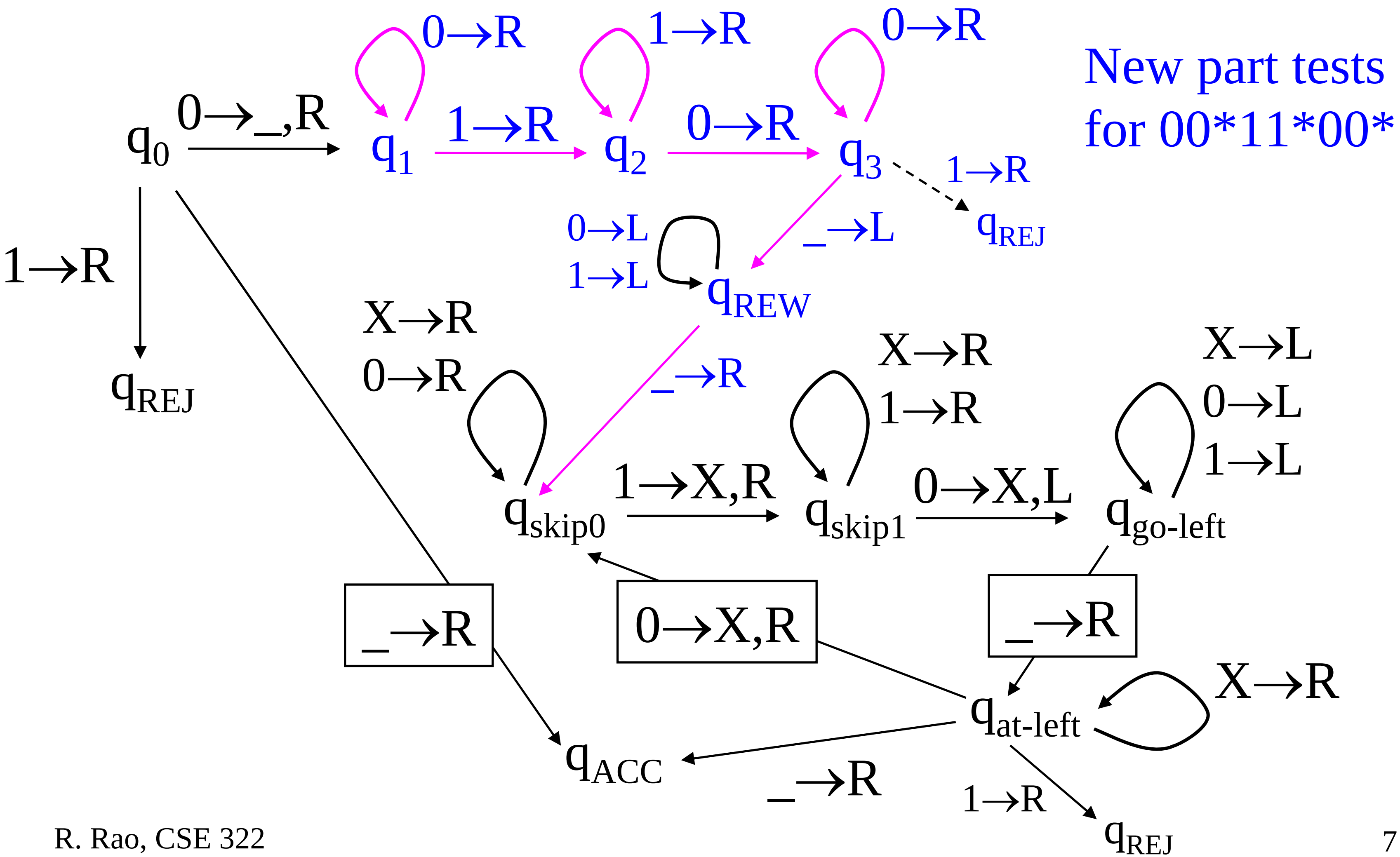
A Simple Fix (to the Decider)

- Scan initially to make sure string is of the form 0*1*0*
- On input w:

Add this

1. If first symbol = blank, ACCEPT
2. If first symbol = 1, REJECT
3. If first symbol = 0: if w is not in 00*11*00*, REJECT; else,
   Write a blank to mark left end of tape
   a. If current symbol is 0 or X, skip until it is 1. REJECT if blank.
   b. Write X over 1. Skip 1's/X's until you see 0. REJECT if blank.
   c. Write X over 0. Move back to left end of tape.
4. At left end: Skip X's until:
   a. You see 0: Write X over 0 and GOTO 3a
   b. You see 1: REJECT
   c. You see a blank space: ACCEPT

The Decider TM for L in all its glory



Can we augment the power of Turing machines with various accessories?

Varieties of TMs

What if we allow multiple tapes?

What if we allow nondeterminism ?

What if my date doesn’t show up tonight?

Various Types of TMs

- **Multi-Tape TMs**: TM with k tapes and k heads
  - $\delta: Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L,R\}^k$
  - $\delta(q_i, a_1, \ldots, a_k) = (q_j, b_1, \ldots, b_k, L, R, \ldots, L)$
- **Nondeterministic TMs (NTMs)**
  - $\delta: Q \times \Gamma \rightarrow Pow(Q \times \Gamma \times \{L,R\})$
  - $\delta(q_i, a) = \{(q_1, b, R), (q_2, c, L), \ldots, (q_m, d, R)\}$
- **Enumerator TM for L**: Prints all strings in L (in any order, possibly with repetitions) and only the strings in L
- Other types: TM with Two-way infinite tape, TM with multiple heads on a single tape, 2D infinite tape TM, Random Access Memory (RAM) TM, etc.

Surprise!
All TMs are born equal…

- Each of the preceding TMs is equivalent to the standard TM
  - They recognize the same set of languages (the Turing-recognizable languages)
- Proof idea: Simulate the “deviant” TM using a standard TM
- Example 1: Multi-tape TM on a standard TM
  - Represent k tapes sequentially on 1 tape using separators #
  - Use new symbol $\underline{a}$ to denote a head currently on symbol *a*

0 1 . . . .

b a h . . . .

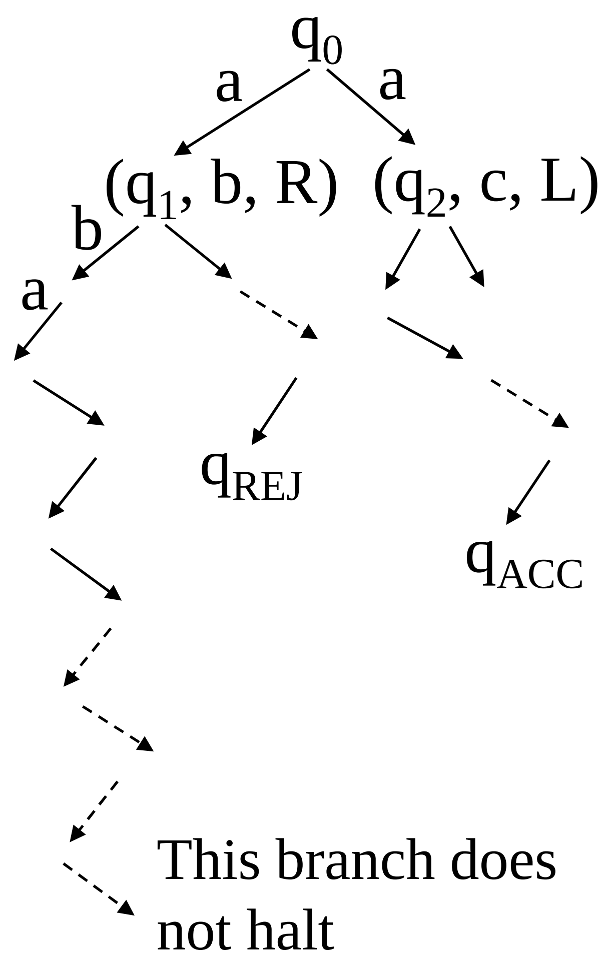
3 2 2 . . . .

$\equiv$ # 0 $\underline{1}$ # b $\underline{a}$ h # $\underline{3}$ 2 2 # . . . .

(See text for details)

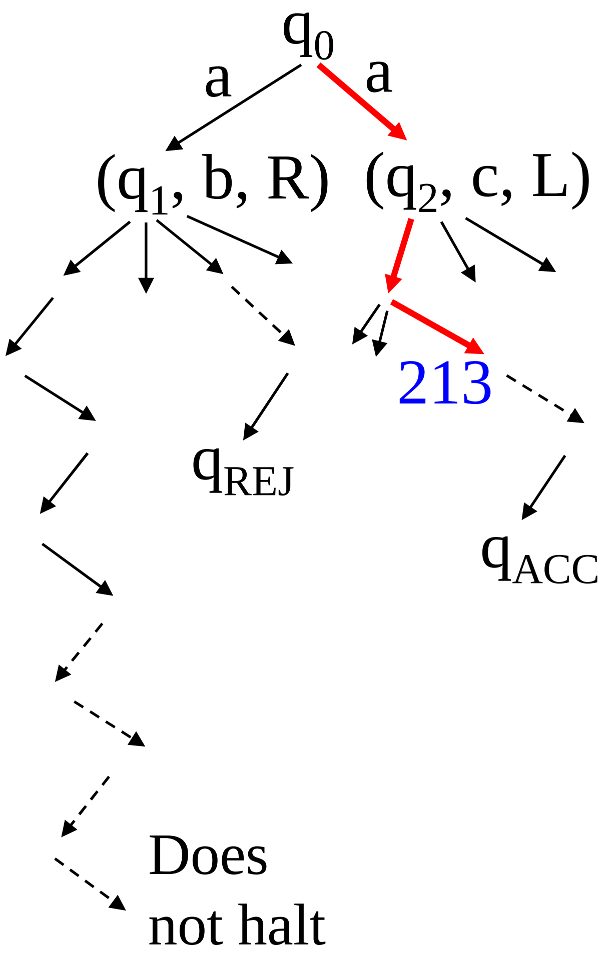
Example 2: Simulating Nondeterminism

- Any nondeterministic TM N can be simulated by a deterministic TM M
  - NTMs: δ: $Q \times \Gamma \rightarrow Pow(Q \times \Gamma \times \{L,R\})$
  - No ε transitions but can simulate them by reading and writing same symbol
  - N accepts w iff there is at least 1 path in N's tree for w ending in q_{ACC}
- **General proof idea**: Simulate each branch sequentially
  - Proof idea 1: Use depth first search? No, might go deep into an infinite branch and never explore other branches!
  - Proof idea 2: Use breadth first search Explore all branches at depth *n* before *n+1*



Simulating Nondeterminism: Details, Details

- Use a 3-tape DTM M for breadth-first traversal of N's tree on w:
  - Tape 1 keeps the input string w
  - Tape 2 stores N's tape during simulation along 1 path (given by tape 3) up to a particular depth, starting with w
  - Tape 3 stores current path number
    E.g. ε = root node q_0
    213 = path made up of 3^{rd} child of 1^{st} child of 2^{nd} child of root
- See text for more details



What about other types of computing machines?

- Enumerator TMs (or Printer Machines)
- TMs with 2-Way Infinite Tape
- TMs with Multiple Read/Write Heads
- TMs with 2-Dimensional Tape
- TMs with Random Access Memory (RAM)

The Church-Turing Thesis

- Various definitions of “algorithms” were shown to be equivalent in the 1930s
- **Church-Turing Thesis**: “The intuitive notion of algorithms equals Turing machine algorithms”
  - Turing machines serve as a precise formal model for the intuitive notion of an algorithm
- “Any computation on a digital computer is equivalent to computation in a Turing machine”

Recap: Recognizable versus Decidable Languages

- A language L is called Turing-Recognizable if there exists a TM M such that L(M) = L
  - Note: M need not halt on all inputs but it should halt and accept all and only those strings that are in L; it can reject strings by either going to q_{rej} or by looping forever
- A TM is a decider if it halts on all inputs
- A language L is decidable if there exists a *decider* D such that L(D) = L

Closure Properties of Decidable Languages

- Decidable languages are closed under $\cup$, $\circ$, $*$, $\cap$, and complement
- Example: Closure under $\cup$
- Need to show that union of 2 decidable L's is also decidable

  Let M1 be a decider for L1 and M2 a decider for L2

  A decider M for L1 $\cup$ L2:

  On input w:

  1. Simulate M1 on w. If M1 accepts, then ACCEPT w. Otherwise, go to step 2 (because M1 has halted and rejected w)
  2. Simulate M2 on w. If M2 accepts, ACCEPT w else REJECT w.

  M accepts w iff M1 accepts w OR M2 accepts w

  i.e. L(M) = L1 $\cup$ L2

Closure Properties

- Consider the proof for closure under $\cup$

  A decider M for $L1 \cup L2$:

  On input w:

  1. Simulate M1 on w. If M1 accepts, then ACCEPT w. Otherwise, go to step 2 (because M1 has halted and rejected w)
  2. Simulate M2 on w. If M2 accepts, ACCEPT w else REJECT w.

  M accepts w iff M1 accepts w OR M2 accepts w

  i.e. $L(M) = L1 \cup L2$

Will the same proof work for showing <u>Turing-recognizable languages</u> are closed under $\cup$? Why/Why not?

M1 may never halt but
w may be in L2

Closure Properties of Recognizable Languages

- Turing recognizable languages are closed under $\cup$

  A TM M for $L1 \cup L2$:

  On input w:

  Simulate M1 and M2 *alternatively* on w step by step.

  If either accepts, then ACCEPT w.

  If both halt and reject w, then REJECT w.

  $L(M) = L1 \cup L2$

  If either M1 or M2 accepts, then M accepts w (even if one of them loops, M will accept and halt when the other accepts and halts because M alternates between M1 and M2). Otherwise, M rejects w by halting or by looping forever.

Closure for Recognizable Languages

- Turing-Recognizable languages are closed under $\cup$, $\circ$, $*$, and $\cap$ (but not complement! We will see this later)
- Example: Closure under $\cap$

  Let M1 be a TM for L1 and M2 a TM for L2 (both may loop)

  A TM M for L1 $\cap$ L2:

  On input w:

  1. Simulate M1 on w. If M1 halts and accepts w, go to step 2. If M1 halts and rejects w, then REJECT w. (If M1 loops, then M will also loop and thus reject w)
  2. Simulate M2 on w. If M2 halts and accepts, ACCEPT w. If M2 halts and rejects, then REJECT w. (If M2 loops, then M will also loop and thus reject w)

  M accepts w iff M1 accepts w AND M2 accepts w i.e. L(M) = L1$\cap$L2