Pumping Lemma

$\forall$ regular language L

$\exists p \quad \forall w \in L \quad |w| \geq p \Longrightarrow$

$\exists\, x, y, z \in \Sigma^*$ s.t.

$w = xyz$

$y \neq \varepsilon$

$|xy| \leq p$

$\forall i \geq 0 \quad xy^iz \in L$

Proof

L is reg, ∴ ∃ DFA $M = (Q, \Sigma, \delta, q_0, F)$ st $L = L(M)$. Let $p = |Q|$. Let w be any $\in L$. if $|w| < p$, vacuous. if $|w| \geq p$ let $r_0\ r_1 \ldots r_{\cancel{|w|}\, p}$ be states entered after reading 0 letters, 1 letter, … $\cancel{|w|}\ p$ letter of w. These $p+1$ states ru that ($i \neq$

By Pigeon hole principle, $\exists\, i < j$ st $r_i = r_j$. Let $x = 1^{st}$ i letters of w, $y = (i+1)^{st}$ through j^{th} letters $z =$ rest.

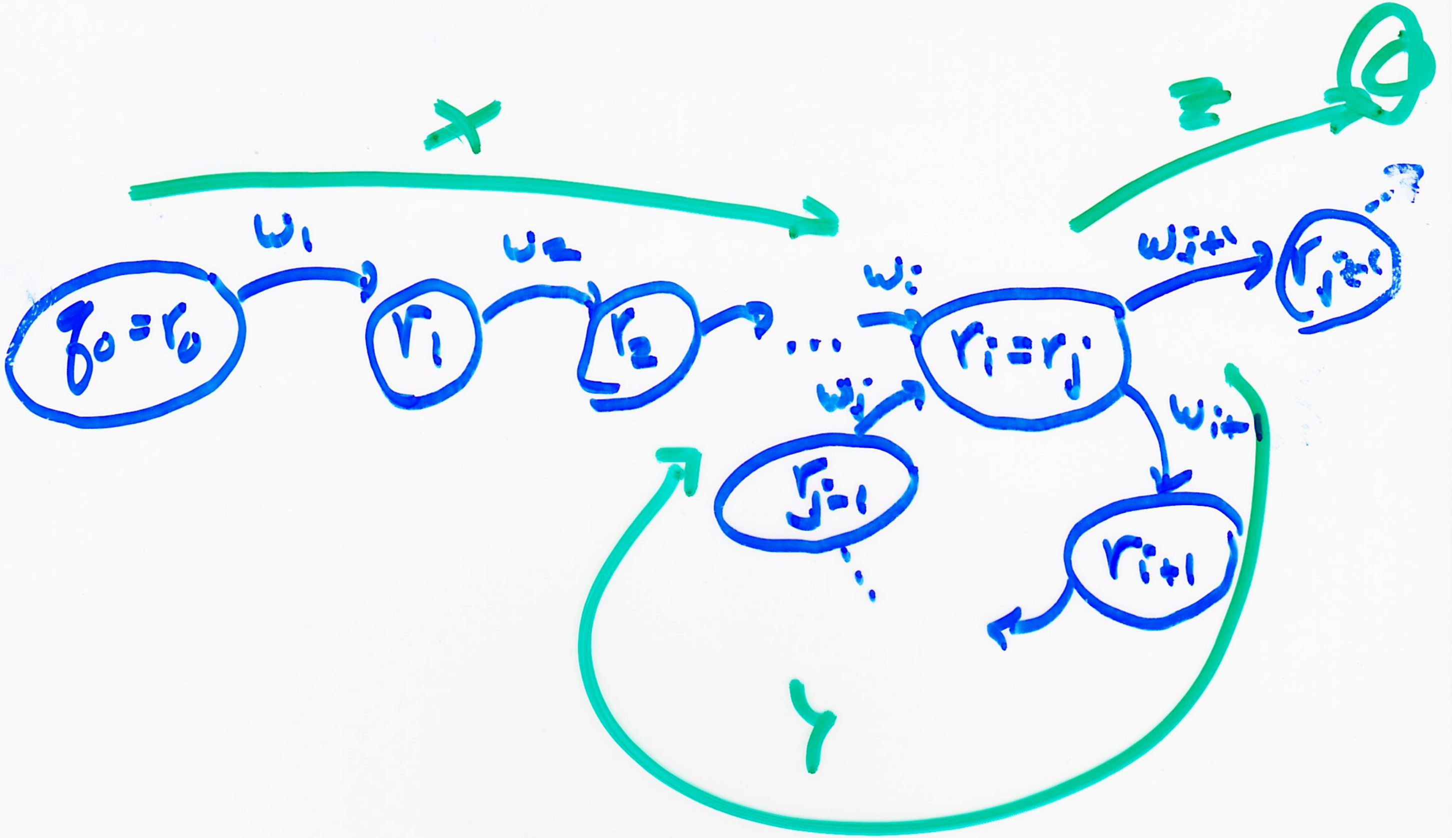
M accepts xz

M accepts xyz

M accepts xy^2z

$\vdots$

M accepts $xy^kz \quad \forall k \geq 0$



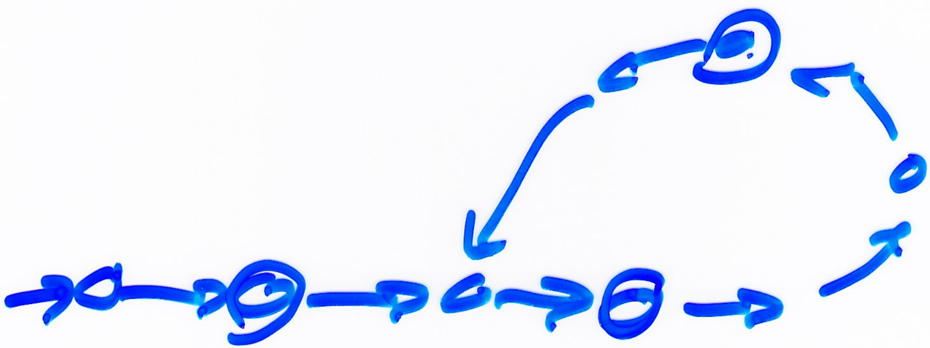
x
z
$q_0 = r_0$
w_1
r_1
w_2
r_2
...
w_i
$r_i = r_j$
w_{j+1}
r_{j+1}
w_j
r_{j-1}
w_{i+1}
r_{i+1}
y

$L = \{ a^{n^2} \mid n \geq 0 \}$ $\quad \Sigma = \{a\}$

Key Idea: perfect squares become increasingly sparse, but PL => at most p gap between members

0 1 4 9 16 25 36

1 3 5 7 9 11



$L = \{ a^{n^2} \mid n \geq 0 \}$ $\Sigma = \{a\}$

Suppose L is regular. By P.L.

$\exists p$ ... let $w = a^{p^2}$ by P.L.

$\exists xyz$ st $w = xyz$

$0 < |y| \leq p$

$$xy^2z = a^{p^2 + |y|}$$

$$(p+1)^2 = p^2 + 2p + 1$$

$$p^2 + |y| \leq p^2 + p < p^2 + 2p + 1$$

$\therefore xy^2z \notin L$

$L = \{ a^n b^n \mid n \geq 0 \}$

if L is regular then by P.L.

$\exists p$ st ...

$w = a^p b^p$

$\exists\ x, y, z \in \Sigma^*$

st $xyz = w$

$|y| > 0$

$|xy| \leq p$

$x = a^i$ for some $0 \leq i < p$

$y = a^j$ for some $1 \leq j \leq p$

$z = a^{p-i-j} b^p$

$x y^2 z = a^{p+j} b^p \notin L$

$\therefore$ L is not regular.

$L = \{ w \mid \#_a(w) = \#_b(w) \}$

$L \cap \underbrace{a^* b^*}_{\text{regular}} = \{ \underbrace{a^n b^n}_{\text{not regular}} \mid n \geq 0 \}$

$\therefore$ By closure of regular languages under intersection, L cannot be regular.