CSE 326: Data Structures
Disjoint Sets ADT

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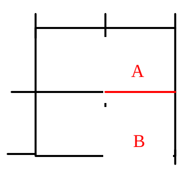
What’s a Good Maze?

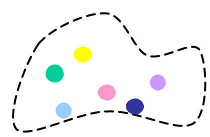
Maze Construction Algorithm

- Given:
  - A collection of rooms **V**
  - Connections between the rooms (initially all closed) **E**
- We want to build a collection of connections to knock down, **E’** Í **E**, such that one unique path connects every two rooms

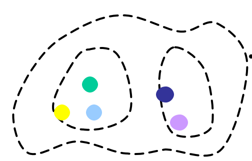
```
While edges remain in E {
  (A, B) = RemoveRandomWall()
  if( A and B have not been
          connected ) {
     Add (A, B) to E’
     Mark A and B as connected
  }
}
```



The Problem, Formally



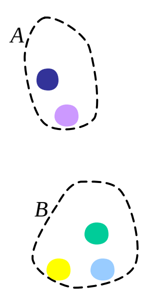
- “If **A** and **B** have not yet been connected”
  - Are two elements in the same set?



- “Mark **A** and **B** as connected”
  - Form the *union* of two sets

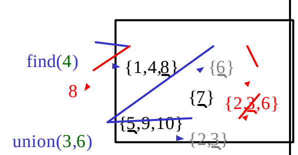
Disjoint Sets ADT

- Find(*x*)
  - Returns set identifier
  - Find(*x*) = Find(*y*) iff *x* and *y* are in the same set
- Union(*A*, *B*)
  - Arguments are set identifiers
  - How do we union the sets containing *x* and *y*?
- MakeNewSet(*item*)
  - Create a new set containing only *item*



Disjoint Sets Formal Properties

- Equivalence property
  - Every element of a DS belongs to exactly one set
- *Dynamic* equivalence property
  - The set of an element can change after execution of a union

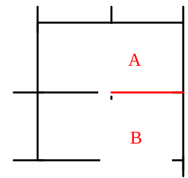


Disjoint Sets Even More Formally

- Given a set $U = \{a_1, a_2, \ldots, a_n\}$
- Maintain a *partition* of U, a set of subsets of U $\{S_1, S_2, \ldots, S_k\}$ such that:
  - Each pair of subsets S_i and S_j are disjoint: $S_i \cap S_j = \varnothing$
  - Together, the subsets cover U: $U = \bigcup_{i=1}^{k} S_i$
  - Each subset has a unique name

Our Modified Maze Construction Algorithm

```
While edges remain in E
  (A, B) = RemoveRandomWall()
  if( Find(A) != Find(B) )
     E' = E' U (A, B)
     Union( Find(A), Find(B) )
```

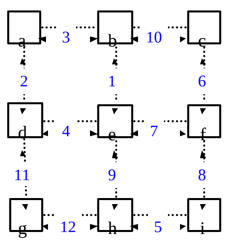


Example

Construct the maze on the right

Initially (the name of each set is underlined):

{a}{b}{c}{d}{e}{f}{g}{h}{i}



Order of edges in blue

Example, continued

{a}{b}{c}{d}{e}{f}{g}{h}{i}

find(b) $\Rightarrow$ b

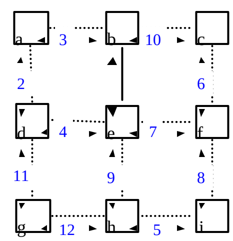
find(e) $\Rightarrow$ e

find(b) $\neq$ find(e) so:

add 1 to **E'**

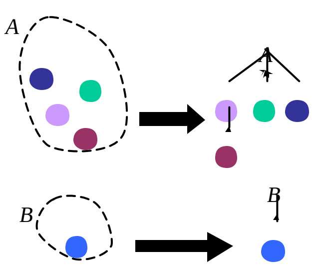
union(b, e)

{a}{b,e}{c}{d}{f}{g}{h}{i}



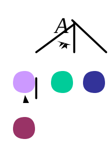
Order of edges in blue

DS ADT Tree Representation



- Maintain a forest of up-trees
- Each set is a tree
- The root of a tree is the set identifier

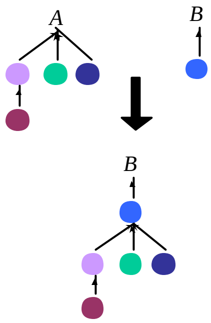
Find Implementation



Find(x)
- Walk parents of x to the root

Runtime:

Union Implementation

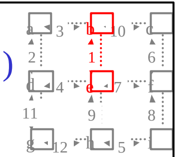


Union(A, B)

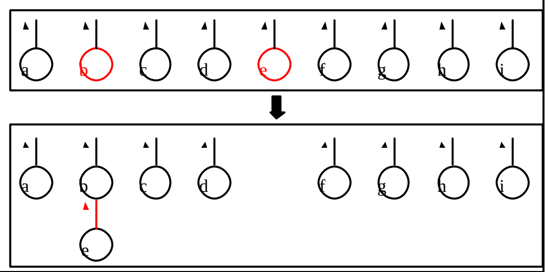
- Join the two trees
- Since A and B are already the roots of a tree, this is easy!

Runtime:

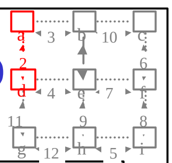
The Whole Example (1/11)



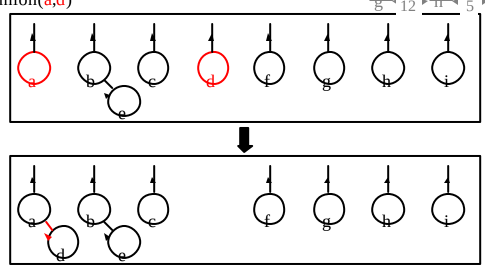
union(b,e)



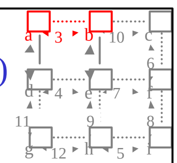
The Whole Example (2/11)



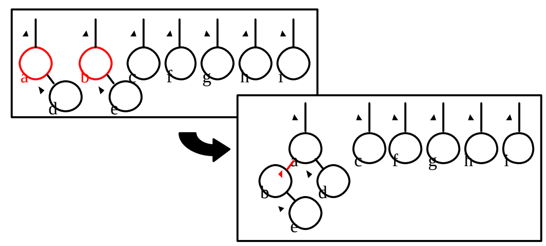
union(a,d)



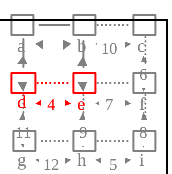
The Whole Example (3/11)



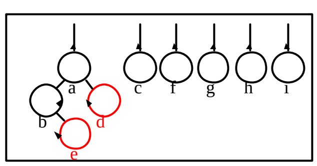
union(a,b)



The Whole Example (4/11)

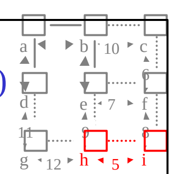


find(d) = find(e)
No union!

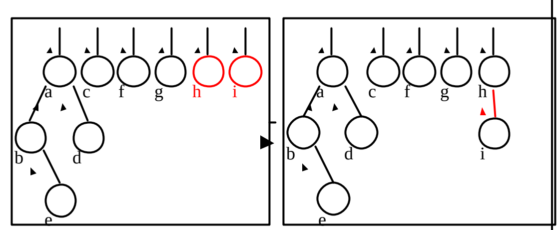


While we're finding ***e****, could we do anything else?*

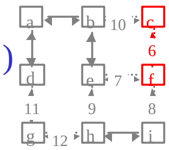
The Whole Example (5/11)



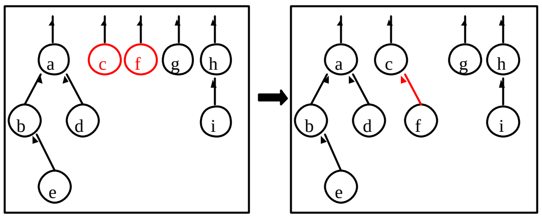
union(h,i)



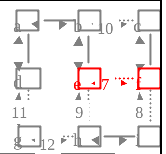
The Whole Example (6/11)



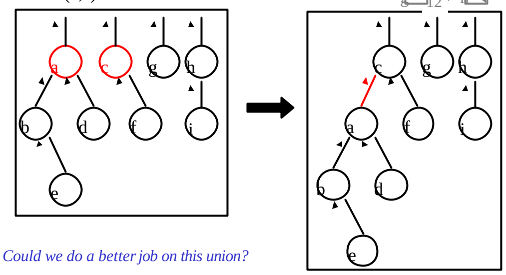
union(c,f)



The Whole Example (7/11)

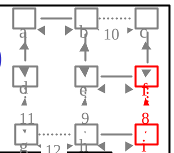


find(e)
find(f)
union(a,c)

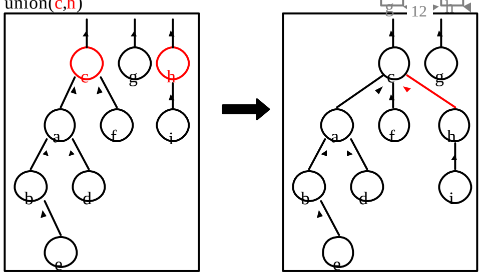


Could we do a better job on this union?

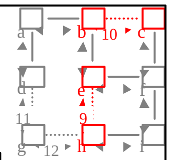
The Whole Example (8/11)



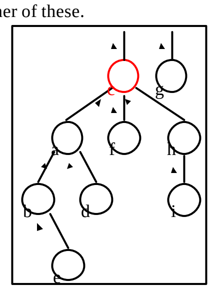
find(f)
find(i)
union(c,h)



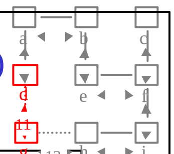
The Whole Example (9/11)



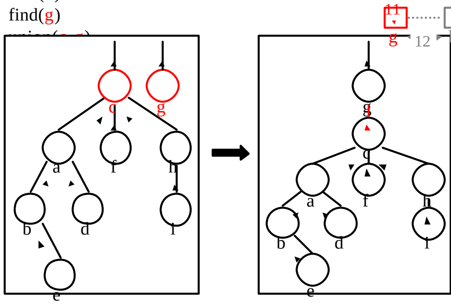
find(e) = find(h) and find(b) = find(c)
So, no unions for either of these.



The Whole Example (10/11)

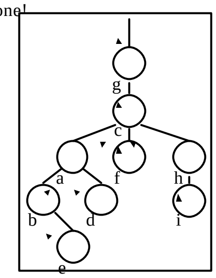


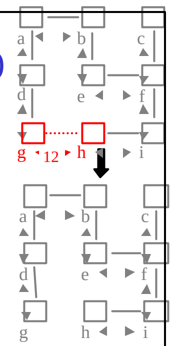
find(d)
find(g)



The Whole Example (11/11)

find(g) = find(h)
So, no union.
And, we're done!



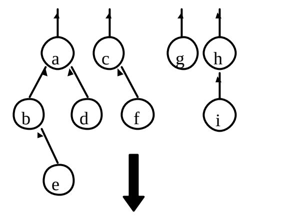


Ooh… scary!
Such a hard maze!

Nifty storage trick

A forest of up-trees can easily be stored in an array.

Also, if the node names are integers or characters, we can use a very simple, perfect hash.



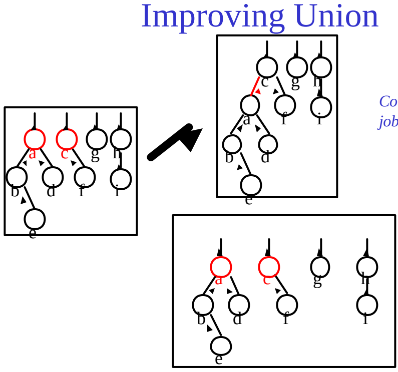
	0 (a)	1 (b)	2 (c)	3 (d)	4 (e)	5 (f)	6 (g)	7 (h)	8 (i)
up-index:	-1	0	-1	0	1	2	-1	-1	7

Implementation

```
typedef ID int;

ID Find(Object x) {
  ID xID = hTable[x];

  while(up[xID] != -1) {
    xID = up[xID];
  }

  return xID;
}
```

```
ID Union(ID x, ID y) {
  up[y] = x;
}
```

Improving Union



Could we do a better job on this union?

Weighted Union Code

```
ID Union(ID x, ID y) {
  // If up[x] and up[y] aren't both
  // -1, this algorithm is in trouble

  if (weight[x] > weight[y]) {
    up[y] = x;
    weight[x] += weight[y];
  }
  else {
    up[x] = y;
    weight[y] += weight[x];
  }
}
```

new runtime for Union():

new runtime for Find():

Weighted Union Find Analysis

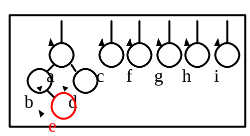
- Finds with weighted union are O(max up-tree height)
- But, an up-tree of height h with weighted union must have at least 2^h nodes

Base case: $h = 0$, tree has $2^0 = 1$ node
Induction hypothesis: assume true for $h < h'$

A merge can only increase tree height by one over the smaller tree. So, a tree of height h'-1 was merged with a larger tree to form the new tree. Each tree then has $\geq 2^{h'-1}$ nodes by the induction hypotheses for a total of at least $2^{h'}$ nodes. QED.

- $\therefore$, $2^{\text{max height}}$ = n and max height = log n
- So, find takes O(log n)

Improving Find

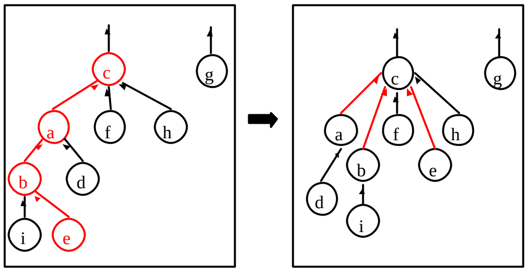


Wait - what's there to improve?

*While we're finding **e**, could we do anything else?*

Path Compression!

find(e)



Path Compression Code

```
ID Find(Object x) {
   // x had better be in
   // the set!
  ID xID = hTable[x];
  ID i = xID;

   // Get the root for
   // this set
   while(up[xID] != -1) {
     xID = up[xID];
   }

   // Change the parent for
   // all nodes along the path
   while(up[i] != -1) {
        temp = up[i];
        up[i] = xID;
        i = temp;
   }
  return xID;
}
```

(New?) runtime for Find():

Interlude: A Tour of Slow Functions

	2	64	1024	32768	2^{20}	2^{30}	$2^{2^{20}}$	$2^{2^{2^{20}}}$
log	1	6	10	15	20	30	2^{20}	$2^{2^{20}}$
log log	0	2.6	3.9	4.6	4.9	4.9	20	2^{20}
log log log	0	1.4	1.9	2.1	2.3	2.3	4.3	20
log*	1	3	3	4	4	4	5	6

Let $\log^{(k)} n = \underbrace{\log (\log (\log \ldots (\log n}_{k \text{ times}})))$

Then, let $\log^* n$ = minimum k such that $\log^{(k)} n \leq 1$

An Even Slower Function

Ackermann created a really big function A(x, y) with the inverse α(x, y) which is really small

How fast does α(x, y) grow?

α(x, y) = 4 for *x* **far** larger than the number of atoms in the universe (2^{300})

α shows up in:

- Computation Geometry (surface complexity)
- Combinatorics of sequences

Complex Complexity of Weighted Union + Path Compression

Tarjan proved that *m* weighted union and find operations on a set of *n* elements have worst case complexity of $O(m \cdot \alpha(m, n))$

For **all** practical purposes this is amortized constant time: $O(m \cdot 4)$ for *m* operations!

In some practical cases, one or both is unnecessary, because trees do not naturally get very deep.

Disjoint Sets ADT Summary

- Also known as Union-Find or Disjoint Set Union/Find
- Simple, efficient implementation
  - With weighted union and path compression
- Great asymptotic bounds
- Kind of weird at first glance, but lots of applications