CSE 326: Data Structures
Worst Case, Average Case, and In-Between

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Today’s Outline

- Review of probability
- Motivation for randomization
- Two randomized data structures
  - Treaps
  - Randomized Skip Lists
- Two randomized algorithms
  - Primality checking
  - Graph searching
    - Again?!

The Problem with Deterministic Data Structures

We’ve seen many data structures with good average case performance on random inputs, but bad behavior on particular inputs

We define the *worst case* runtime over all possible inputs *I* of size *n* as:

$$\text{Worst-case } T(n) = \max_{I} T(I)$$

We define the *average case* runtime over all possible inputs *I* of size *n* as:

$$\text{Average-case } T(n) = \left(\sum_{I} T(I) \right) / \mathit{numPossInputs}$$

The Motivation for Randomization

Instead of randomizing the input (since we cannot!), consider randomizing the data structure

- No bad inputs, just unlucky random numbers
- Expected case good behavior on any input

Expectant Cases

Definition:

- A *worst-case expected time* analysis is a *weighted sum* of all possible outcomes over some probability distribution

Thus, the *expected* runtime of a *randomized* data structure on some input *I* is:

$$\text{Expected } T(I) = \sum_{S} (\Pr(S) * T(I, S))$$

And the *worst-case expected* runtime of a *randomized* data structure* is:

$$\text{Expected } T(n) = \max_{I} \left(\sum_{S} (\Pr(S) * T(I, S)) \right)$$

* Randomized data structure = = a data structure whose behaviour is dependant on a sequence of random numbers

What’s the Difference?

- Randomized with good expected time
  - Once in a while you will have an expensive operation, but no inputs can make this happen all the time
- Deterministic with good average time
  - If your application happens to always use the “bad” case, you are in big trouble!
- *Expected time is kind of like an insurance policy for your algorithm!*

Allstate.
You're in good hands.

Does Randomization Work?

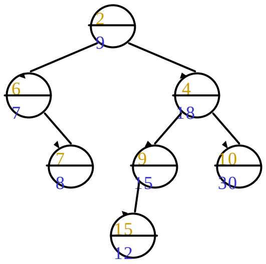
- Not-really randomized data structures
  - Splay Trees (compare with AVL trees)
  - Skew Heaps (compare with Leftist heaps)
  - Others?
- Randomized data structures
  - Universal Hashing hash table
    - Also Perfect Hashing hash table
  - Others?

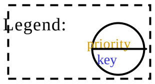
Treap Data Structure for the Dictionary ADT

Treaps:

- Have the binary tree *structure* property
- Have the BST *order* property
- Have the heap *order* property with randomly assigned priorities

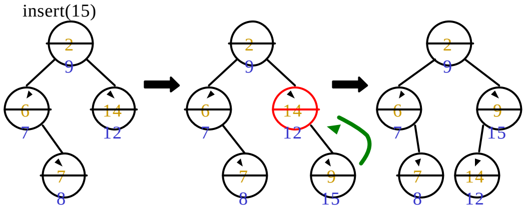
heap in yellow; search tree in blue





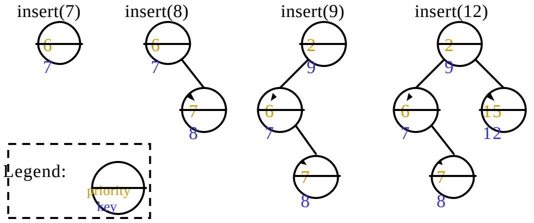
Treap Insert

- Choose a random priority
- Insert as in normal BST
- Rotate up until heap order is restored (maintaining BST property while rotating)



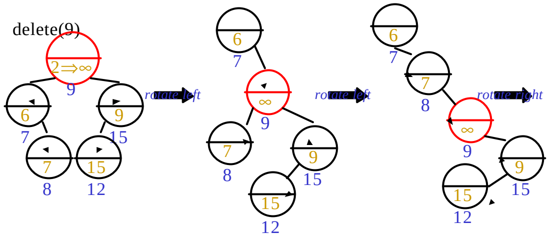
Tree + Heap… Why Bother?

Insert data in sorted order into a treap; what shape tree comes out?

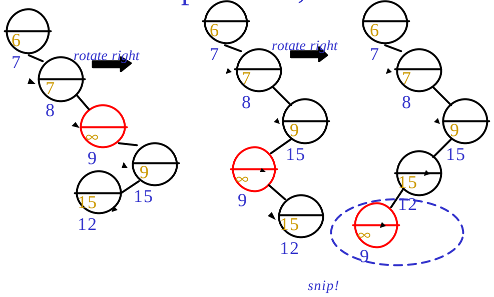


Treap Delete

- Find the key
- Increase its priority to ∞
- Rotate it to the fringe
- Snip it off



Treap Delete, cont.



Treap Summary

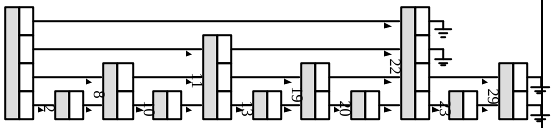
Implements Dictionary ADT
  - Insert in *expected* O(log n) time
  - Delete in *expected* O(log n) time
  - Find in *expected* O(log n) time
  - But *worst* case O(n)

Memory use
  - O(1) per node
  - About the cost of AVL trees

Very simple to implement, little overhead
  - Less than AVL trees

Perfect Binary Skip List

- Sorted linked list
- # of links of a node is its *height*
- The height *i* link of each node (that has one) links to the next node of height *i* or greater



Find() in a Perfect Binary Skip List

- Start *i* at the maximum height
- Until the node is found, or *i* =1 and the next node is too large:
  - If the next node along the *i* link is less than the target, traverse to the next node
  - Otherwise, decrease *i* by one

Runtime?

Insert() in a Perfect Binary Skip List

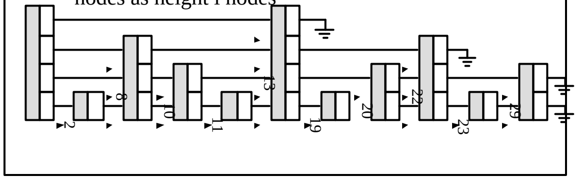
Runtime?

Randomized Skip List Intuition

- It's *far* too hard to insert into a perfect skip list
- But is perfection necessary?
- What matters in a skip list?
  - We want way fewer tall nodes than short ones
  - Make good progress through the list with each high traverse

Randomized Skip List

- Sorted linked list
- # of links of a node is its height
- The height i link of each node (that has one) links to the next node of height i or greater
- There should be *about 1/2 as many* height i+1 nodes as height i nodes



Find() in a RSL

- Start *i* at the maximum height
- Until the node is found or *i* is one and the next node is too large:
  - If the next node along the *i* link is less than the target, traverse to the next node
  - Otherwise, decrease *i* by one

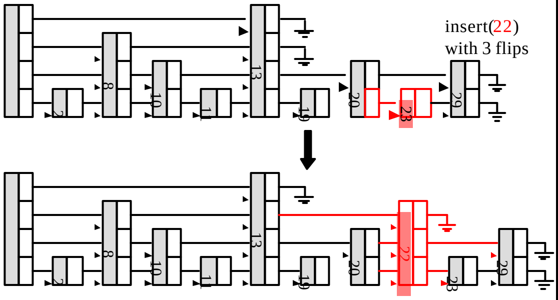
Same as for a perfect skip list!

Runtime?

Insert() in a RSL

- Flip a coin until it comes up heads
  - This will take i flips. Make the new node’s height i.
- Do a find, remembering nodes where we moved down one link
- Add the new node at the spot where the find ends
- Point all the nodes where we moved down (up to the new node’s height) at the new node
- Point the new node’s links where those redirected pointers were pointing

RSL Insert Example



insert(22)
with 3 flips

Runtime?

Iteration and1D Range Queries

- Iteration: successively return (in order) each element in the structure
  - Start at beginning, walk list to end
  - Just like a linked list!
- Range query: search for everything that falls between two values
  - Find() start point
  - Walk through skip list using the lowest links
  - Output each node until the end point

Runtimes?

Randomized Skip List

Summary

- Implements Dictionary ADT
  - Insert in *expected* O(log n)
  - Find in *expected* O(log n)
  - But *worst* case O(n)
- Memory use
  - O(1) memory per node
  - About double a linked list
- Less overhead than search trees for iteration over range queries

Intermission: Efficiently Calculating Powers

- How would you implement a function/method `pow(x, n)` which returns the number x^n?
- How could you do that *efficiently*?

Primality Checking

- Given a number *P*, can we determine whether or not *P* is prime?

```
Date: Wed, 7 Aug 2002 11:00:43 -0700 (PDT)
Newsgroups: uw-cs.ugrads.openforum
Subject: Primes in P??

So, a paper published yeterday alleges they have found
   a deterministic polynomial algorithm to determine
   primality.

http://www.cse.iitk.ac.in/primality.pdf
```

Two Properties of Primes

`P` is a prime **`0 < A < P`** and **`0 < X < P`**

Then:

$$A^{P-1} = 1 \pmod{P}$$

And, the only solutions to $X^2 = 1 \pmod{P}$ are:

$X = 1$ and $X = P - 1$

Checking Primality - First Attempt

```
aToPMinus1 =
   pow( someNumber, p-1 );
if( aToPMinus1 % p == 1 )
   return true;
else
   return false;
```

```
int pow(int a, int n) {
   if (n == 0)
       return 1;
   if( n == 1 )
       return a;

   int x = pow( a, n/2 );

   if( isEven(n) )
       return x * x;
   else
       return x * x * a;
}
```

Using More Information

“And, the *only* solutions to $X^2 = 1 \pmod{P}$ are:

$X = 1$ and $X = P - 1$”

Checking Primality - Second Attempt

```
int pow(int a, int n, int p) {
   if (n == 0)
      return 1;
   if (n == 1)
      return a;

   int x = pow( a, n/2, p );

   int squared = x * x % p;
   if( squared == 1 && x is neither p-1 or 1)
      // p isn't prime!

   if( isEven(n) )
      return x * x;
   else
      return x * x * a;
```

Checking Primality

Systematic algorithm:

- For all A such that 0 < A < P
  - Calculate A^{P-1} mod P using `pow()`
  - Check at each step of `pow()` and at end for primality conditions

Randomized algorithm:

- Randomly pick an A and calculate A^{P-1} mod P using `pow()`

If pow() tells us that P is prime, will either algorithm tell us conclusively that P really is prime?

Randomized Primality Check

If the randomized algorithm reports failure, then P really isn't prime.

If the randomized algorithm reports success, then P *might be* prime.
  - P is prime with probability > ¾
  - Each new A has independent probability of false positive

- Solution:
  - Run the randomized algorithm several times

Evaluating Randomized Primality Testing

Your probability of being struck by lightning this year: 0.00004%

Your probability that a number that tests prime 11 times in a row is actually not prime: 0.00003%

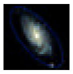
Your probability of winning a lottery of 1 million people five times in a row: 1 in 2^{100}

Your probability that a number that tests prime 50 times in a row is actually not prime: 1 in 2^{100}

Randomized Graph Searching

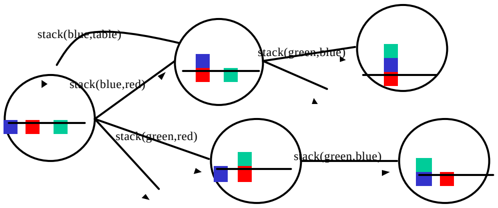
Consider some really huge graphs…
  - All cities and towns in the World Atlas
  - All stars in the Galaxy
  - All ways 10 blocks can be stacked

Huh???



Implicitly Generated Graphs

- A huge graph may be implicitly specified by rules for generating it on-the-fly
- Blocks world:
  - vertex = relative positions of all blocks
  - edge = robot arm stacks one block



The Blocks World Problem: A Large Branching Factor

- Source = initial state of the blocks
- Goal = desired state of the blocks
- Path source to goal = sequence of actions (program) for robot arm!
- n blocks ≈ n^n vertices
  - 10 blocks ≈ 10 billion vertices!
- We *cannot* search such huge graphs exhaustively!
  - Breadth-first search: If out-degree of each node is 10, potentially visits 10^d vertices
  - Dijkstra's algorithm is basically breadth-first search (modified to handle edge weights)

Review: Heuristic-Based Searching

- The *Manhattan distance* (Δ x+ Δ y) is an estimate of the distance to the goal
  - A heuristic value
- Best-First Search
  - Select nodes to *minimize estimated distance to the goal*; if a promising set of nodes doesn't pan out, *backtrack*
- Hill-climbing
  - Select nodes to *minimize estimated distance to the goal*
  - Says nothing about backtracking. In fact, we don't even keep track of our previous path!

“Hill Climbing”: What’s in a Name?

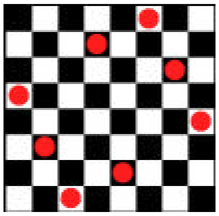
- Let’s assume we’re trying to *maximize* our heuristic
- If we view our graph as a terrain, and the heuristic values as elevations, then graph searching becomes a problem of finding the tallest hill
- But …
  - What are some problems with hill climbing?

Solution: Hill Climbing with *Random Restarts**

- Once you can’t go any farther, randomly choose a node in the graph, and try again.
- Once you have several possible solutions, pick the one with the highest heuristic value
- A common variation is *simulated annealing*, which may pick a random move instead of the best move. As the algorithm progresses, we choose the random move less and less often

Example: N-Queens Problem

- Place N queens on an N by N chessboard so that no two queens can attack each other
- Graph search formulation:
  - Each way of placing from 0 to N queens on the chessboard is a vertex
  - Edge between vertices that differ by adding or removing one queen
  - Start vertex: empty board
  - Goal vertex: any one with N non-attacking queens (there are many such goals)



Hill Climbing with Random Restarts – Complexity?

- Can often prove that if you run long enough will reach a goal state – but may take exponential time
- In some cases can prove that a hill-climbing or random walk algorithm will find a goal in polynomial time with high probability
  - e.g., 2-SAT, Papadimitriou 1997
- Widely used for real-world problems where actual complexity is unknown – scheduling, optimization
  - N-Queens – probably polynomial, but no one has tried to prove formal bound

Other Real-World Applications

- Routing finding – computer networks, airline route planning
- VLSI layout – cell layout and channel routing
- Production planning – “just in time” optimization
- Protein sequence alignment
- Travelling Salesman
- Many other “NP -Hard” problems
  - A class of problems for which no exact polynomial time algorithms exist – so heuristic search is the best we can hope for

Other Randomized Algorithms/Data Structures

- Data Structures
  - Other Dictionary ADT’s
- Algorithms
  - Find a minimum spanning tree in $O(E)$ time
  - Max flow problems in $O(V^2 \log V)$ time
    - In CSE 421, you’ll see a deterministic algorithm works in $O(VE^2)$ time
  - Many others!