CSE 326: Data Structures
Asymptotic Analysis

Hannah Tang and Brian Tjaden
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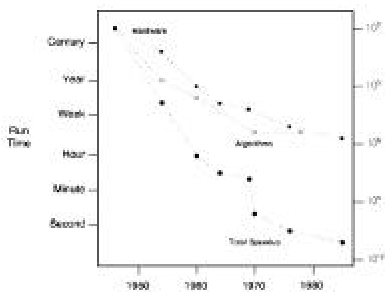
Today's Outline

- How's the project going?
- Finish up stacks, queues, lists, and bears, oh my!
- Math review and runtime analysis
- Pretty pictures
- Asymptotic analysis

Analyzing Algorithms: Why Bother?



From "Programming Pearls", by Jon Bentley
Communications of the ACM, Nov 1984

Analyzing Algorithms

- Computer scientists analyze algorithms to precisely characterize an algorithm's:
  - Time complexity (running time)
  - Space complexity (memory use)
- This allows us to get a better sense of the various tradeoffs between several algorithms
  - For instance, do we know how complex the 1984 algorithm is, compared to the 1945 algorithm?

A problem's input size is indicated by a number n
  - Sometimes have multiple inputs, e.g. m and n

- The running time of an algorithm is a function of n
  - n, 2^n, $n \log n$, $18 + 3n(\log n^2) + 5n^3$

Hannah Takes a Break

```
bool ArrayFind(int array[],
               int n,
               int key )
{
  // Insert your algorithm
  here
}
```

2	3	5	16	37	50	73	75	126

What algorithm would you choose to implement this code snippet?

Hannah Takes a Break: Simplifying assumptions

- Ideal single-processor machine (serialized operations)
- "Standard" instruction set (load, add, store, etc.)
- All operations take 1 time unit (including, for our purposes, each Java or C++ statement

HTaB: Analyzing Code

Basic Java/C++ operations	Constant time
Consecutive statements	Sum of times
Conditionals	Larger branch plus test
Loops	Sum of iterations
Function calls	Cost of function body
Recursive functions	Solve recurrence relation

HTaB: Linear Search Analysis

```
bool ArrayFind(   int array[],
                  int n,
                  int key )
{

  for( int i = 0; i < n; i++ )
  {
     // Found it!
     if( array[i] == key )
          return true;
  }
  return false;

}
```

- Exact Runtime:
- Best Case:
- Worst Case:

HTaB: Binary Search Analysis

```
bool ArrayFind( int array[], int s,
                int e, int key ) {
  // The subarray is empty
  if( e – s <= 0 )
     return false;

  // Search this subarray
  int mid = (e - s) / 2;
  if( array[key] == array[mid] ) {
     return true;
  } else if( key < array[mid] ) {
     return ArrayFind( array, s,
                       mid, key );
  } else {
     return ArrayFind( array, mid,
                       e, key );
}
```

- Exact Runtime:
- Best case:
- Worst case:

Back to work: Solving Recurrence Relations

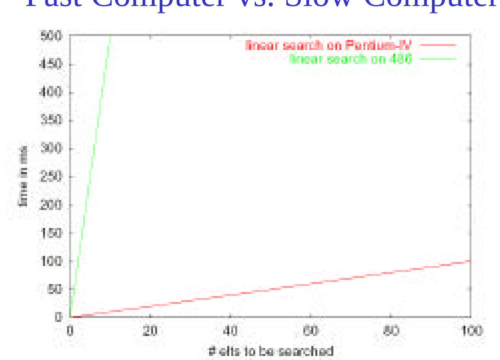
1. Determine the recurrence relation. What are the base case(s)?
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case

Linear Search vs Binary Search

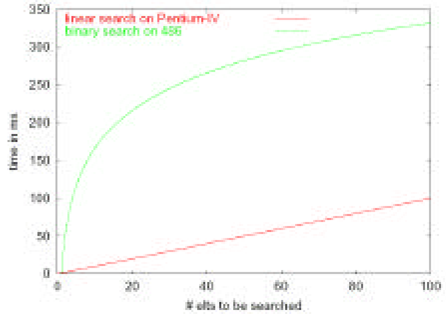
	Linear Search	Binary Search
Exact Runtime		
Best Case		
Worst Case		

So … which algorithm is best?
What the tradeoffs did you make?

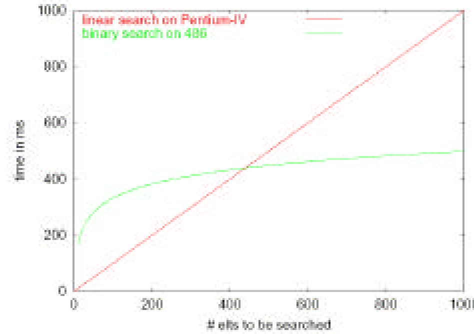
Fast Computer vs. Slow Computer



Fast Computer vs. Smart Programmer (round 1)



Fast Computer vs. Smart Programmer (round 2)



Asymptotic Analysis

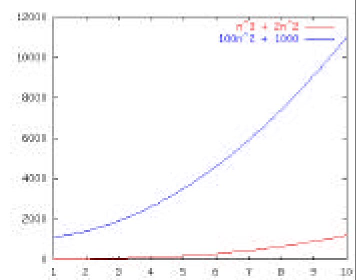
- Asymptotic analysis looks at the *order* of the running time of the algorithm
  - A valuable tool when the input gets "large"
  - Ignores the *effects of different machines* or *different implementations* of the same algorithm
- Intuitively, to find the asymptotic runtime, throw away the constants and low-order terms
  - Linear search is $T(n) = n \in O(n)$
  - Binary search is $T(n) = T(n) = 4 \log_2 n + 1 \in O(\log n)$

Remember: the fastest algorithm has the slowest growing function for its runtime

Order Notation: Intuition

$f(n) = n^3 + 2n^2$

$g(n) = 100n^2 + 1000$



Although not yet apparent, as n gets "sufficiently large", $f(n)$ will be "greater than or equal to" $g(n)$

Order Notation: Definition

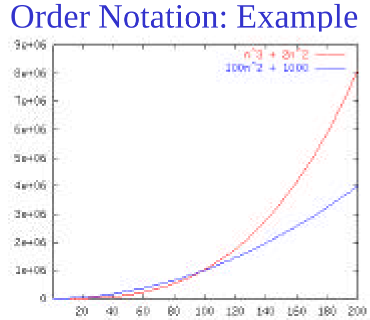
O(f(n)) is a *set of functions*

$g(n) \in O(f(n))$ iff
There exist c and n_0 such that $g(n) \leq c\, f(n)$ for all $n \geq n_0$

Example:
$100n^2 + 1000 \leq 5\,(n^3 + 2n^2)$ for all $n \geq 19$
So $g(n) \in O(f(n))$

Sometimes, you'll see the notation $g(n) = O(f(n))$. This equivalent to $g(n) \in O(f(n))$. However, the notation $O(f(n)) = g(n)$ is *not* correct

Order Notation: Example

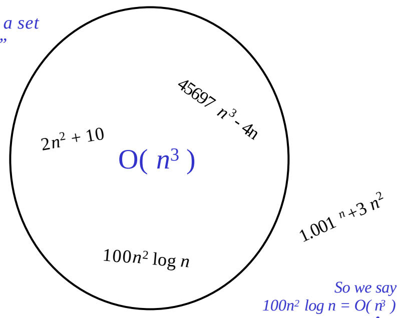


$100n^2 + 1000 \leq 5\,(n^3 + 2n^2)$ for all $n \geq 19$

So $g(n) \in O(f(n))$

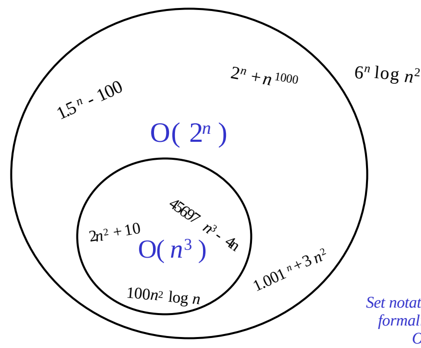
Oops: Set Notation

"O(f(n)) is a set of functions"



So we say both
$100n^2 \log n = O(n^3)$ and
$100n^2 \log n$ Î $O(n^3)$

Set Notation



Set notation allows us to formalize our intuition
$O(n^3) \subset O(2^n)$

Big-O Common Names

constant:	O(1)	
logarithmic:	O(log n)	
linear:	O(n)	
log-linear:	O(n log n)	
superlinear:	O(n^{1+c}) (c is a constant, where 0 < c < 1)	
quadratic:	O(n^2)	
polynomial:	O(n^k)	(k is a constant)
exponential:	O(c^n)	(c is a constant > 1)

Meet the Family

- O(f(n)) is the set of all functions asymptotically less than or equal to f(n)
  - o(f(n)) is the set of all functions asymptotically strictly less than f(n)
- Ω(f(n)) is the set of all functions asymptotically greater than or equal to f(n)
  - ω(f(n)) is the set of all functions asymptotically strictly greater than f(n)
- θ(f(n)) is the set of all functions asymptotically equal to f(n)

Meet the Family Formally
(don't worry about dressing up)

- g(n) ∈ O(f(n)) iff
  There exist c and n_0 such that g(n) £ c f(n) for all $n \geq n_0$
  - g(n) ∈ o(f(n)) iff
    There exists a n_0 such that g(n) < c f(n) for all c and $n \geq n_0$
- g(n) ∈ Ω(f(n)) iff
  There exist c and n_0 such that g(n) ^{3} c f(n) for all $n \geq n_0$
  - g(n) ∈ ω(f(n)) iff
    There exists a n_0 such that g(n) > c f(n) for all c and $n \geq n_0$
- g(n) ∈ θ(f(n)) iff
  g(n) ∈ O(f(n)) and g(n) ∈ Ω(f(n))

Big-Omega et al. Intuitively

Asymptotic Notation	Mathematics Relation
O	≤
Ω	≥
θ	=
o	<
ω	>

True or False?

$10{,}000\,n^2 + 25n \in \theta(n^2)$	
$10^{-10}\,n^2 \in \theta(n^2)$	
$n^3 + 4 \in \omega(n^2)$	
$n \log n \in O(2^n)$	
$n \log n \in \Omega(n)$	
$n^3 + 4 \in o(n^4)$	

Another Kind of Analysis

- Runtime may depend on actual input, not just length of input
- Analysis based on input type:
  - Worst case
    - Your worst enemy is choosing input
  - Average case
    - Assume a probability distribution of inputs
  - Best case
    - Not too useful
- Amortized analysis
    - Runtime over many runs, regardless of underlying probability for inputs

HTaB: Pros and Cons of Asymptotic Analysis

To Do

- Start project 1
  - Due Monday, July 1st at 10 PM sharp!
- Sign up for 326 mailing list(s)
  - Don't forget to use the new web interfaces!
- Prepare for tomorrow's quiz
  - Possible topics:
    - Math concepts from 321 (skim section 1.2 in Weiss)
    - Lists, stacks, queues, and the tradeoffs between various implementations
    - Whatever asymptotic analysis stuff we covered today
    - Possible middle names for Brian C. Tjaden, Hannah C. Tang, and Albert J. Wong
- Read chapter 2 (algorithm analysis), section 4.1 (introduction to trees), and sections 6.1-6.4 (priority queues and binary heaps)