CSE 326: Data Structures

Graphs

Neva Cherniavsky
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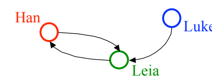
Graph… ADT?

- Not quite an ADT… operations not clear
- A formalism for representing relationships between objects
  Graph `G = (V,E)`
  - *Set of vertices*: $V = \{v_1, v_2, \ldots, v_n\}$
  - *Set of edges*: $E = \{e_1, e_2, \ldots, e_m\}$ where each e_i connects two vertices (v_{i1}, v_{i2})



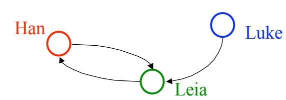

```
V = {Han, Leia, Luke}
E = {(Luke, Leia),
     (Han, Leia),
     (Leia, Han)}
```

Graphs In Practice

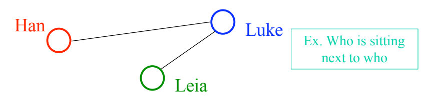
- Web graph
  - Vertices are web pages
  - Edge from u to v is a link to v appears on u
- Call graph of a computer program
  - Vertices are functions
  - Edge from u to v is u calls v
- Task graph for a work flow
  - Vertices are tasks
  - Edge from u to v if u must be completed before v begins

Graph Definitions

In *directed* graphs, edges have a specific direction:



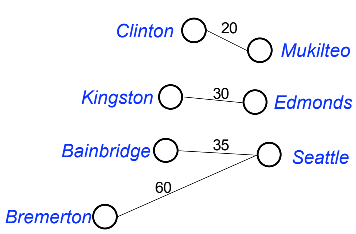
In *undirected* graphs, they don't (edges are two-way):



v is *adjacent* to u if $(u,v) \in E$

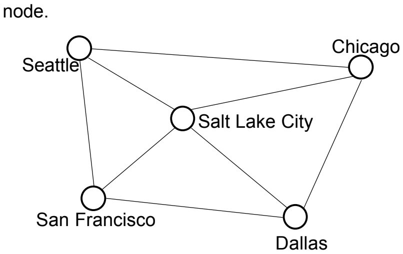
Weighted Graphs

Each edge has an associated weight or cost.



Paths and Cycles

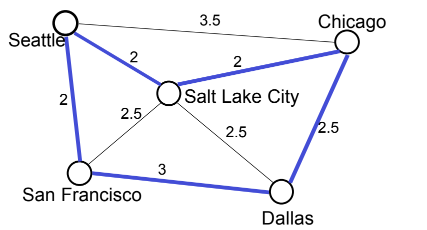
- A *path* is a list of vertices $\{v_1, v_2, \ldots, v_n\}$ such that $(v_i, v_{i+1}) \in E$ for all $0 \le i < n$.
- A *cycle* is a path that begins and ends at the same node.



- *p = {Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle}*

Path Length and Cost

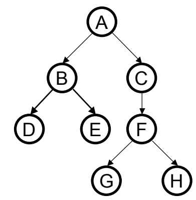
- *Path length*: the number of edges in the path
- *Path cost*: the sum of the costs of each edge



length(p) = 5 cost(p) = 11.5

Trees as Graphs

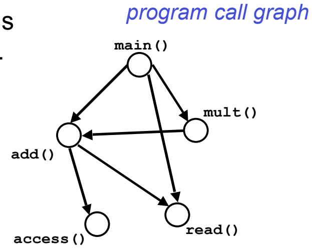
- Every tree is a graph with some restrictions:
  - › the tree is *directed*
  - › there are *no cycles* (directed or undirected)
  - › there is a *directed path from the root to every node*



Directed Acyclic Graphs (DAGs)

- DAGs are directed graphs with no cycles.

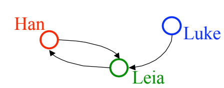
program call graph



Trees ⊂ DAGs ⊂ Graphs

Graph Representation 1: Adjacency Matrix

•A `|V| x |V|` array in which an element `(u, v)` is true if and only if there is an edge from `u` to `v`



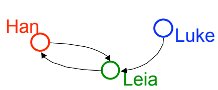
	Han	Luke	Leia
Han			
Luke			
Leia			

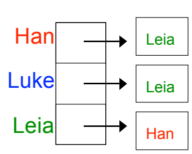
Runtime:
iterate over vertices?
iterate over edges?
iterate edges adj. to vertex?
edge exists?

Space required?

Graph Representation 2: Adjacency List

- A `|V|`-ary list (array) in which each entry stores a list (linked list) of all adjacent vertices

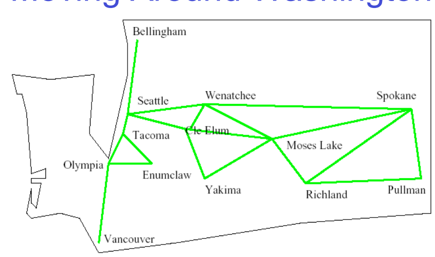




Runtime:
iterate over vertices?
iterate over edges?
iterate edges adj. to vertex?
edge exists?

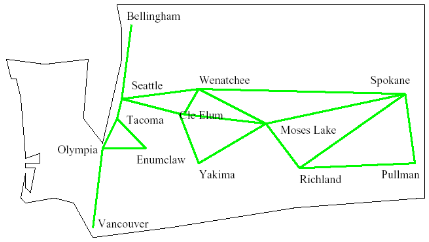
Space required?

Some Applications: Moving Around Washington



What's the *shortest way* to get from Seattle to Pullman?
Edge labels:

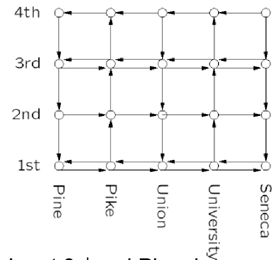
Some Applications: Moving Around Washington



What's the *fastest way* to get from Seattle to Pullman?

Edge labels:

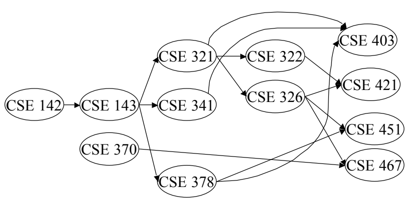
Some Applications: Bus Routes in Downtown Seattle



If we're at 3rd and Pine, how can we get to 1st and University using Metro?

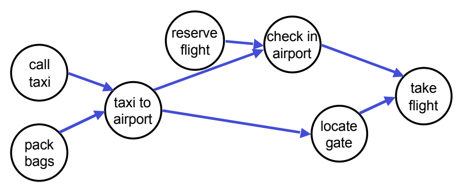
Application: Topological Sort

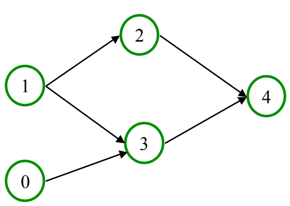
Given a directed graph, **G = (V,E)**, output all the vertices in **V** such that no vertex is output **before** any other vertex with an edge to it.



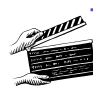
Is the output unique?

Another example





Valid Topological Sorts:



Topological Sort: Take One

1. Label each vertex with its *in-degree* (# of inbound edges)
2. **While** there are vertices remaining:
   a. Choose a vertex *v* of *in-degree zero*; output *v*
   b. Reduce the in-degree of all vertices adjacent to *v*
   c. Remove *v* from the list of vertices

Runtime:

```
void Graph::topsort(){
  Vertex v, w;

  labelEachVertexWithItsIn-degree();

  for (int counter=0; counter < NUM_VERTICES;
                                   counter++){
    v = findNewVertexOfDegreeZero();

    v.topologicalNum = counter;
    for each w adjacent to v
      w.indegree--;
  }
}
```

Time? (labelEachVertexWithItsIn-degree)

Time? (findNewVertexOfDegreeZero)

Time? (for each w adjacent to v)

Where is the bottleneck?

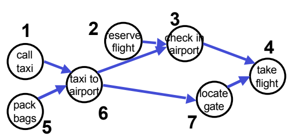
Topological Sort: Take Two

1. Label each vertex with its in-degree
2. Initialize a queue Q to contain all in-degree zero vertices
3. While Q not empty
   a. v = Q.dequeue; output v
   b. Reduce the in-degree of all vertices adjacent to v
   c. If new in-degree of any such vertex u is zero Q.enqueue(u)

Note: could use a stack, list, set, box, … instead of a queue

Runtime:

Example

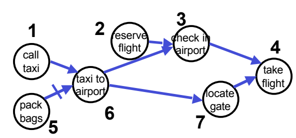


In-degree

1	0	→ 6
2	0	→ 3
3	2	→ 4
4	2	
5	0	→ 6
6	2	→ 3 → 7
7	1	→ 4

Queue = 1 2 5
Output =

Example

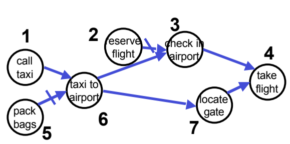


In-degree

1	0	→ 6
2	0	→ 3
3	2	→ 4
4	2	
5	0	→ 6
6	1	→ 3 → 7
7	1	→ 4

Queue = 1 2
Output = 5

Example

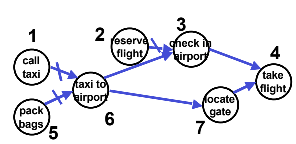


In-degree

1	0	→ 6
2	0	→ 3
3	1	→ 4
4	2	
5	0	→ 6
6	1	→ 3 → 7
7	1	→ 4

Queue = 1
Output = 5 2

Example

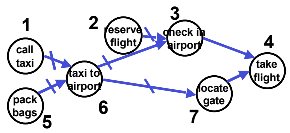


In-degree

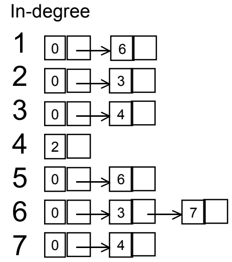
1	0	→ 6
2	0	→ 3
3	1	→ 4
4	2	
5	0	→ 6
6	0	→ 3 → 7
7	1	→ 4

Queue = 6
Output = 5 2 1

Example

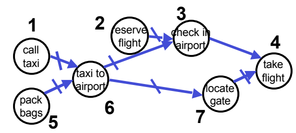


In-degree

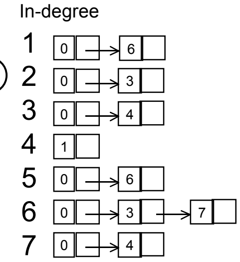


Queue = 3 7
Output = 5 2 1 6

Example

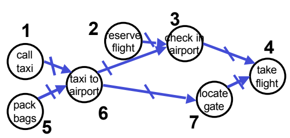


In-degree

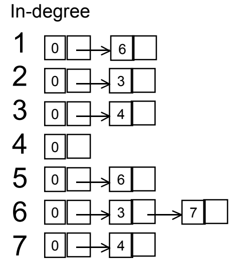


Queue = 3
Output = 5 2 1 6 7

Example



In-degree



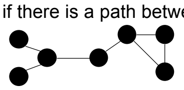
Queue = 4
Output = 5 2 1 6 7 3

Exercise

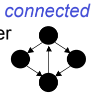
- Design the algorithm to initialize the in-degree array. Assume the adjacency list representation.

Graph Connectivity

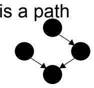
Undirected graphs are *connected* if there is a path between any two vertices



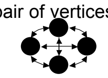
Directed graphs are *strongly connected* if there is a path from any one vertex to any other



Directed graphs are *weakly connected* if there is a path between any two vertices, *ignoring direction*



A *complete* graph has an edge between every pair of vertices



Graph Traversals

- Breadth-first search (and depth-first search) work for arbitrary (directed or undirected) graphs - not just mazes!
  - Must mark visited vertices so you do not go into an infinite loop!
- Either can be used to determine connectivity:
  - Is there a path between two given vertices?
  - Is the graph (weakly) connected?
- Which one:
  - Uses a queue?
  - Uses a stack?
  - Always finds the shortest path (for unweighted graphs)?