CSE 326: Data Structures

Shortest Path

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Announcements

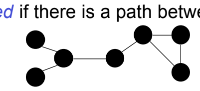
- Project 3 code due tomorrow
- Project 3 readme and benchmarking due next week

```
void Graph::topsort(){
  Queue q(NUM_VERTICES);  int counter = 0; Vertex v, w;
  labelEachVertexWithItsIn-degree();
                                                    Time?
  q.makeEmpty();
  for each vertex v
    if (v.indegree == 0)
      q.enqueue(v);                       Time?

  while (!q.isEmpty()){
    v = q.dequeue();
    v.topologicalNum = ++counter;
    for each w adjacent to v           Time?
      if (--w.indegree == 0)
        q.enqueue(w);
  }
}
```

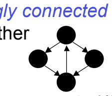
Runtime:

Graph Connectivity

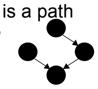
Undirected graphs are *connected* if there is a path between any two vertices



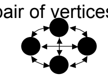
Directed graphs are *strongly connected* if there is a path from any one vertex to any other



Directed graphs are *weakly connected* if there is a path between any two vertices, *ignoring direction*



A *complete* graph has an edge between every pair of vertices



Graph Traversals

- Breadth-first search (and depth-first search) work for arbitrary (directed or undirected) graphs - not just mazes!
  - Must mark visited vertices so you do not go into an infinite loop!
- Either can be used to determine connectivity:
  - Is there a path between two given vertices?
  - Is the graph (weakly) connected?
- Which one:
  - Uses a queue?
  - Uses a stack?
  - Always finds the shortest path (for unweighted graphs)?

The Shortest Path Problem

Given a graph *G*, edge costs $c_{i,j}$, and vertices *s* and *t* in *G*, find the shortest path from *s* to *t*.

For a path $p = v_0\ v_1\ v_2 \ldots v_k$

- *unweighted length* of path $p = k$ (a.k.a. *length*)
- *weighted length* of path $p = \sum_{i=0..k-1} c_{i,i+1}$ (a.k.a *cost*)

Path length equals path cost when ?

Single Source Shortest Paths (SSSP)

Given a graph *G*, edge costs $c_{i,j}$, and vertex *s*, find the shortest paths from *s* to all vertices in G.

- Is this harder or easier than the previous problem?

All Pairs Shortest Paths (APSP)

Given a graph *G* and edge costs $c_{i,j}$, find the shortest paths between all pairs of vertices in G.

- Is this harder or easier than SSSP?
- Could we use SSSP as a subroutine to solve this?

Variations of SSSP

- Weighted vs. unweighted
- Directed vs undirected
- Cyclic vs. acyclic
- Positive weights only vs. negative weights allowed
- Shortest path vs. longest path
- …

Applications

- Network routing
- Driving directions
- Cheap flight tickets
- Critical paths in project management (see textbook)
- …

SSSP: Unweighted Version

Ideas?

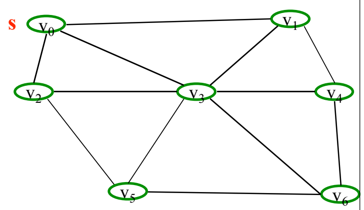
```
void Graph::unweighted (Vertex s){
  Queue q(NUM_VERTICES);
  Vertex v, w;
  q.enqueue(s);
  s.dist = 0;

  while (!q.isEmpty()){
    v = q.dequeue();
    for each w adjacent to v
      if (w.dist == INFINITY){
        w.dist = v.dist + 1;
        w.path = v;
        q.enqueue(w);
      }
    }
  }
```

each edge examined at most once – if adjacency lists are used

each vertex enqueued at most once

total running time: O(   )

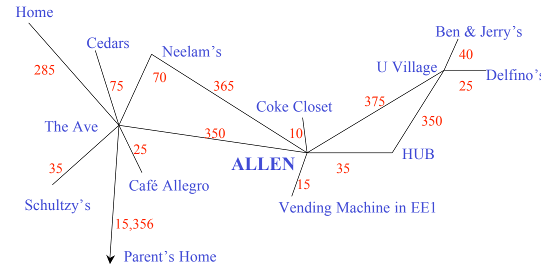
Student Activity



V	Dist	path
v0		
v1		
v2		
v3		
v4		
v5		
v6		

Queue

Weighted SSSP: The Quest For Food



Can we calculate shortest distance to all nodes from Allen Center?

Dijkstra, Edsger Wybe

Legendary figure in computer science; was a professor at University of Texas.

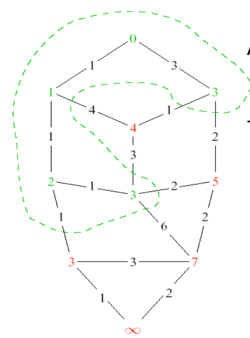
Supported teaching introductory computer courses without computers (pencil and paper programming)

Supposedly wouldn't (until very late in life) read his e-mail; so, his staff had to print out messages and put them in his box.

E.W. Dijkstra (1930-2002)

1972 Turning Award Winner,
Programming Languages, semaphores, and …

Dijkstra's Algorithm: Idea

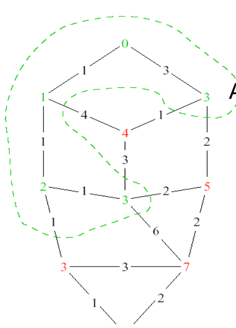


Adapt BFS to handle weighted graphs

Two kinds of vertices:

- Finished or known vertices
  - Shortest distance has been computed
- Unknown vertices
  - Have tentative distance

Dijkstra's Algorithm: Idea



At each step:

1) Pick closest unknown vertex
2) Add it to known vertices
3) Update distances

Dijkstra's Algorithm: Idea

Similar to breadth-first search, but uses a heap instead of a queue:

- Always select (expand) the vertex that has a lowest-cost path to the start vertex

Correctly handles the case where the lowest-cost (shortest) path to a vertex is not the one with fewest edges

Important Features

- Once a vertex is removed from the head, the cost of the shortest path to that node is known
- While a vertex is still in the heap, another shorter path to it might still be found
- The shortest path itself can found by following the backward pointers stored in node.previous

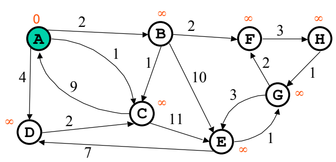
Dijkstra's Algorithm: Pseudocode

Initialize the cost of each node to ∞

Initialize the cost of the source to 0

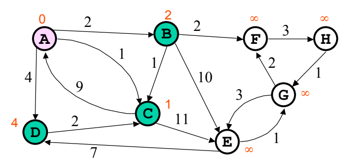
While there are unknown nodes left in the graph
- Select an unknown node *b* with the lowest cost
- Mark *b* as known
- For each node *a* adjacent to *b*
  - *a*'s cost = min(*a*'s old cost, *b*'s cost + cost of (*b*, *a*))

Dijkstra's Algorithm in Action



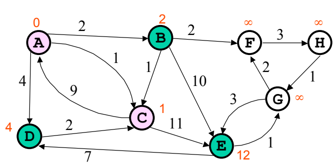
vertex	visited	cost
A		0
B		??
C		??
D		??
E		??
F		??
G		??
H		??

Dijkstra's Algorithm in Action



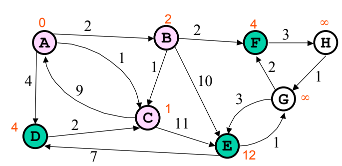
vertex	visited	cost
A	x	0
B		?
C		?
D		?
E		??
F		??
G		??
H		??

Dijkstra's Algorithm in Action



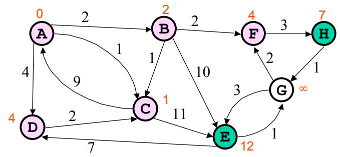
vertex	visited	cost
A	x	0
B		?
C	x	?
D		?
E		??
F		??
G		??
H		??

Dijkstra's Algorithm in Action



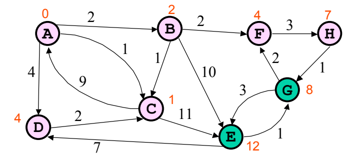
vertex	visited	cost
A	x	0
B	x	?
C	x	?
D		?
E		??
F		?
G		??
H		??

Dijkstra's Algorithm in Action



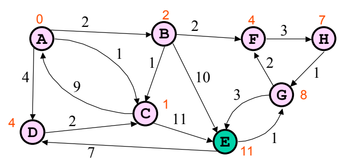
vertex	visited	cost
A	x	0
B	x	?
C	x	?
D	x	?
E		??
F	x	?
G		??
H		?

Dijkstra's Algorithm in Action



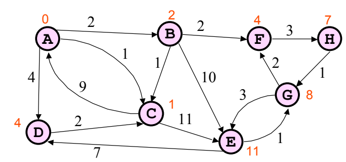
vertex	visited	cost
A	x	0
B	x	?
C	x	?
D	x	?
E		??
F	x	?
G		?
H	x	?

Dijkstra's Algorithm in Action



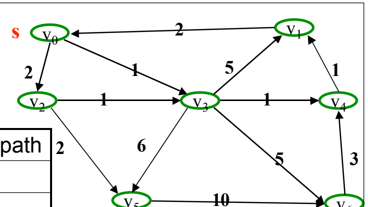
vertex	visited	cost
A	x	0
B	x	?
C	x	?
D	x	?
E		??
F	x	?
G	x	?
H	x	?

Dijkstra's Algorithm in Action



vertex	visited	cost
A	x	0
B	x	?
C	x	?
D	x	?
E	x	??
F	x	?
G	x	?
H	x	?

Student Activity



V	Known	Dist	path
v0			
v1			
v2			
v3			
v4			
v5			
v6			

Dijkstra's Alg: Implementation

Initialize the cost of each node to ∞

Initialize the cost of the source to 0

While there are unknown nodes left in the graph

- Select the unknown node *b* with the lowest cost
- Mark *b* as known
- For each node *a* adjacent to *b*
  - *a*'s cost = min(*a*'s old cost, *b*'s cost + cost of (*b*, *a*))

What data structures should we use?

Running time?

Dijkstra's Algorithm: a Greedy Algorithm

Greedy algorithms always make choices that *currently* seem the best

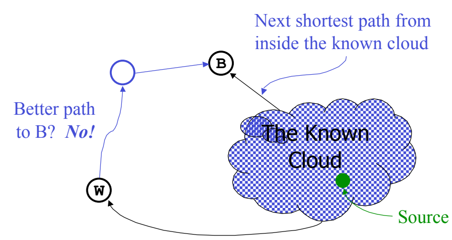
- Short-sighted – no consideration of long-term or global issues
- Locally optimal - does not always mean globally optimal!!

Correctness of Dijkstra's

Intuition for correctness:

- shortest path from source vertex to itself is 0
- cost of going to adjacent nodes is at most edge weights
- cheapest of these must be shortest path to that node
- update paths for new node and continue picking cheapest path

Correctness: The Cloud Proof



How does Dijkstra's decide which vertex to add to the Known set next???

- If path to B is shortest, path to W must be *at least as long*
  (or else we would have picked W as the next vertex)
- So any path *through* W to B *cannot* be any shorter!

Correctness: Inside the Cloud

Prove by induction on # of nodes in the cloud:

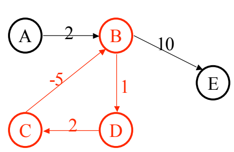
Initial cloud is just the source with shortest path 0

Assume: Everything inside the cloud has the correct shortest path

Inductive step: Only when we prove the shortest path to some node ***v*** (which is *not* in the cloud) is correct, we add it to the cloud

When does Dijkstra's algorithm not work?

The Trouble with Negative Weight Cycles



What's the shortest path from A to E?

Problem?

Dijkstra's vs BFS

At each step:
1) Pick closest unknown vertex
2) Add it to finished vertices
3) Update distances

Dijkstra's Algorithm

At each step:
1) Pick vertex from queue
2) Add it to visited vertices
3) Update queue with neighbors

Breadth-first Search

Some Similarities:

Dijkstra's Algorithm: Summary

- Classic algorithm for solving SSSP in weighted graphs *without negative weights*
- A *greedy* algorithm (irrevocably makes decisions without considering future consequences)
- Intuition for correctness:
  - shortest path from source vertex to itself is 0
  - cost of going to adjacent nodes is at most edge weights
  - cheapest of these must be shortest path to that node
  - update paths for new node and continue picking cheapest path