CSE 326: Data Structures

Minimum Spanning Tree

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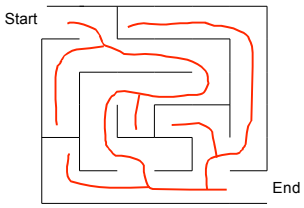
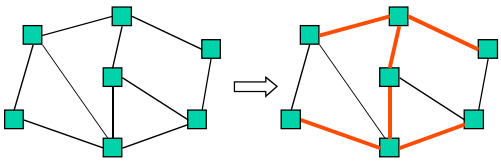
A Hidden Tree



Spanning Tree in a Graph

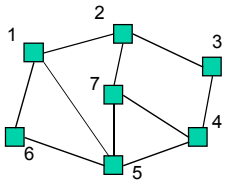


Vertex = router
Edge = link between routers

Spanning tree
- Connects all the vertices
- No cycles

Undirected Graph

- G = (V,E)
  - V is a set of vertices (or nodes)
  - E is a set of unordered pairs of vertices



V = {1,2,3,4,5,6,7}
E = {{1,2},{1,6},{1,5},{2,7},{2,3},
{3,4},{4,7},{4,5},{5,6}}

2 and 3 are adjacent
2 is incident to edge {2,3}

Spanning Tree Problem

- Input: An undirected graph G = (V,E). G is connected.
- Output: T contained in E such that
  - (V,T) is a connected graph
  - (V,T) has no cycles

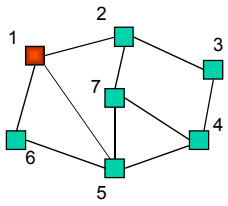
Spanning Tree Algorithm

```
ST(i: vertex)
   mark i;
   for each j adjacent to i do
      if j is unmarked then
         Add {i,j} to T;
         ST(j);
end{ST}
```

```
Main
T := empty set;
ST(1);
end{Main}
```

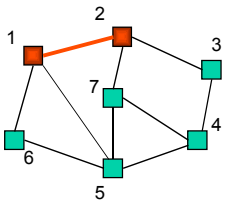
Example of Depth First Search

ST(1)



Example Step 2

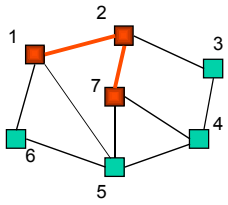
ST(1)
ST(2)



{1,2}

Example Step 3

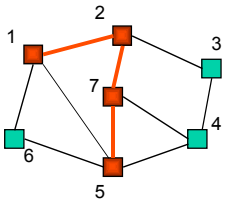
ST(1)
ST(2)
ST(7)



{1,2} {2,7}

Example Step 4

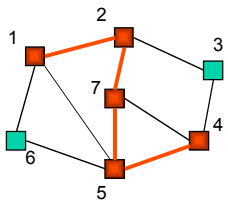
ST(1)
ST(2)
ST(7)
ST(5)



{1,2} {2,7} {7,5}

Example Step 5

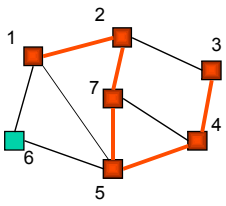
ST(1)
ST(2)
ST(7)
ST(5)
ST(4)



{1,2} {2,7} {7,5} {5,4}

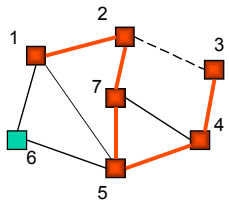
Example Step 6

ST(1)
ST(2)
ST(7)
ST(5)
ST(4)
ST(3)



{1,2} {2,7} {7,5} {5,4} {4,3}

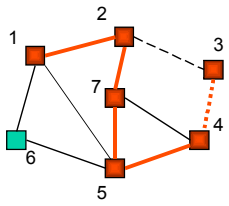
Example Step 7



ST(1)
ST(2)
ST(7)
ST(5)
ST(4)
ST(3)

{1,2} {2,7} {7,5} {5,4} {4,3}

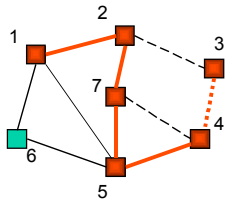
Example Step 8



ST(1)
ST(2)
ST(7)
ST(5)
ST(4)

{1,2} {2,7} {7,5} {5,4} {4,3}

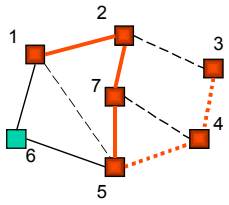
Example Step 9



ST(1)
ST(2)
ST(7)
ST(5)
ST(4)

{1,2} {2,7} {7,5} {5,4} {4,3}

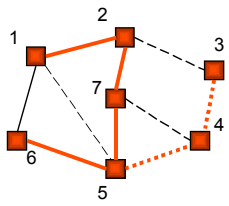
Example Step 10



ST(1)
ST(2)
ST(7)
ST(5)

{1,2} {2,7} {7,5} {5,4} {4,3}

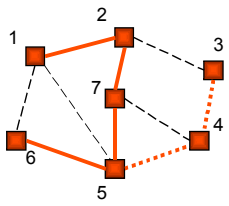
Example Step 11



ST(1)
ST(2)
ST(7)
ST(5)
ST(6)

{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

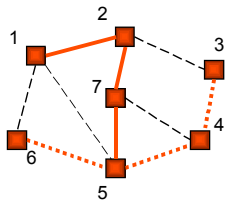
Example Step 12



ST(1)
ST(2)
ST(7)
ST(5)
ST(6)

{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

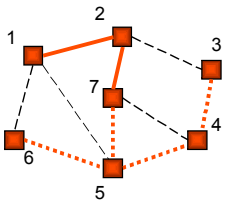
Example Step 13



ST(1)
ST(2)
ST(7)
ST(5)

{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

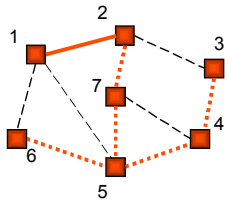
Example Step 14



ST(1)
ST(2)
ST(7)

{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

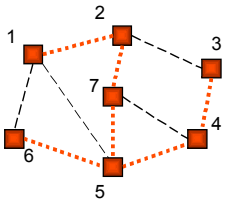
Example Step 15



ST(1)
ST(2)

{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

Example Step 16

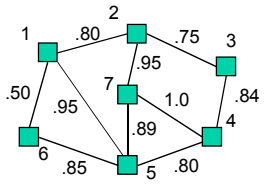


ST(1)

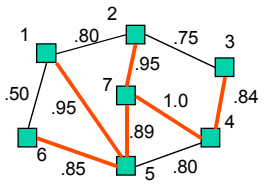
{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

Best Spanning Tree

- Each edge has the probability that it won't fail
- Find the spanning tree that is least likely to fail



Example of a Spanning Tree



Probability of success = .85 x .95 x .89 x .95 x 1.0 x .84
= .5735

Minimum Spanning Trees

Given an undirected graph **G**=(**V**,**E**), find a graph **G'**=(**V**, **E'**) such that:

- E' is a subset of E
- |E'| = |V| - 1
- G' is connected
- $\sum_{(u,v)\in E'} c_{uv}$ is minimal

G' is a **minimum spanning tree**.

Applications: wiring a house, power grids, Internet connections

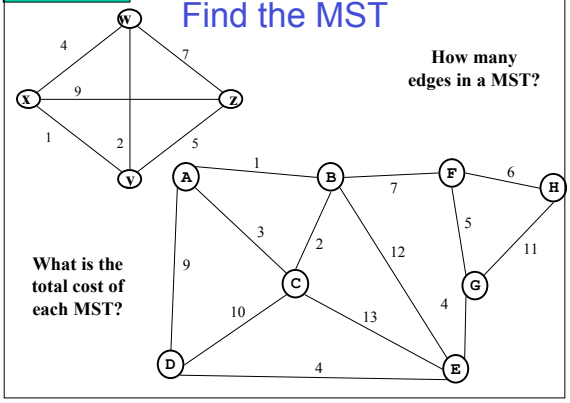
Minimum Spanning Tree Problem

- Input: Undirected Graph G = (V,E) and a cost function C from E to the reals. C(e) is the cost of edge e.
- Output: A spanning tree T with minimum total cost. That is: T that minimizes

$$C(T) = \sum_{e \in T} C(e)$$

Student Activity

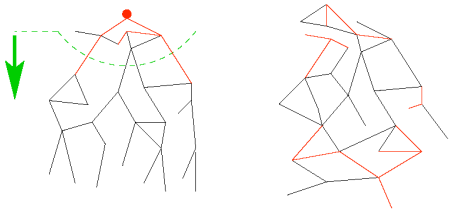
Find the MST



How many edges in a MST?

What is the total cost of each MST?

Two Different Approaches

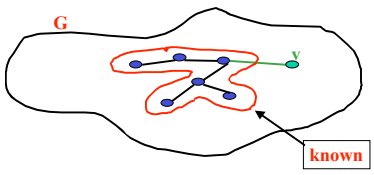


Prim's Algorithm
Almost identical to Dijkstra's

Kruskals's Algorithm
Completely different!

Prim's algorithm

Idea: Grow a tree by adding an edge from the "known" vertices to the "unknown" vertices. Pick the ***edge with the smallest weight.***



Prim's Algorithm for MST

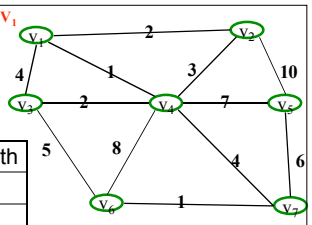
A *node-based* greedy algorithm
Builds MST by greedily adding nodes

1. Select a node to be the "root"
   - mark it as known
   - Update cost of all its neighbors
2. While there are unknown nodes left in the graph
   a. Select an unknown node *b* with the smallest cost from some *known* node *a*
   b. Mark *b* as known
   c. Add (*a*, *b*) to MST
   d. Update cost of all nodes adjacent to *b*

Student Activity

Start with V_1

Find MST using Prim's



V	Kwn	Distance	path
v1			
v2			
v3			
v4			
v5			
v6			
v7			

Order Declared Known:

V_1

Prim's Algorithm Analysis

Running time:

Same as Dijkstra's: O(|E| log |V|)

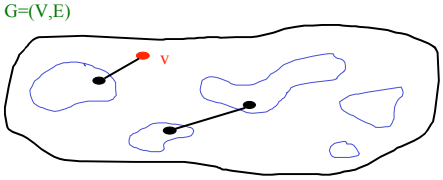
Correctness:

Proof is similar to Dijkstra's

Kruskal's MST Algorithm

Idea: Grow a forest out of edges that do not create a cycle. Pick an ***edge with the smallest weight.***

G=(V,E)



Kruskal's Algorithm for MST

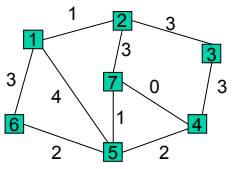
An *edge-based* greedy algorithm

Builds MST by greedily adding edges

1. Initialize with
   - empty MST
   - all vertices marked unconnected
   - all edges unmarked
2. While there are still unmarked edges
   a. Pick the lowest cost edge `(u,v)` and mark it
   b. If `u` and `v` are not already connected, add `(u,v)` to the MST and mark `u` and `v` as connected to each other

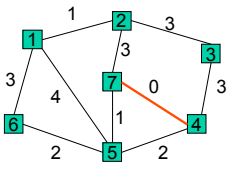
Doesn't it sound familiar?

Example of Kruskal 1



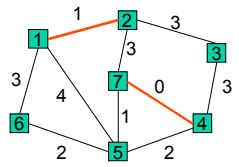
{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

Example of Kruskal 2



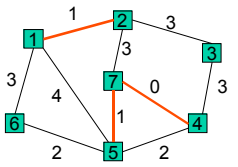
{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

Example of Kruskal 2



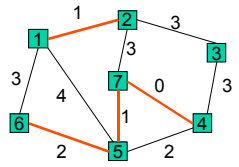
{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

Example of Kruskal 3



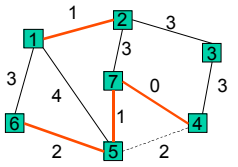
{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

Example of Kruskal 4



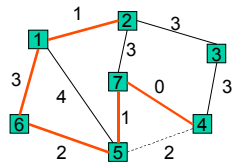
{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

Example of Kruskal 5



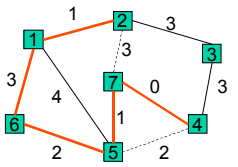
{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

Example of Kruskal 6



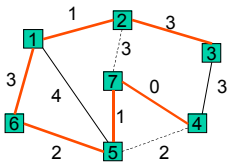
{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

Example of Kruskal 7



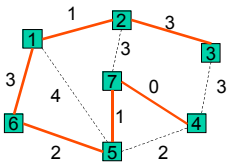
{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

Example of Kruskal 7



{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

Example of Kruskal 8,9



{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

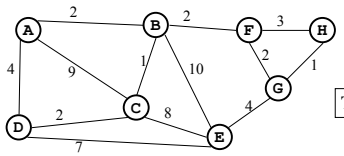
Kruskal code

```
void Graph::kruskal(){
  int edgesAccepted = 0;
  DisjSet s(NUM_VERTICES);

  while (edgesAccepted < NUM_VERTICES - 1){
    e = smallest weight edge not deleted yet;
    // edge e = (u, v)
    uset = s.find(u);
    vset = s.find(v);
    if (uset != vset){
      edgesAccepted++;
      s.unionSets(uset, vset);
    }
  }
}
```

Student Activity

Find MST using Kruskal's



Total Cost:

- Now find the MST using Prim's method.
- Under what conditions will these methods give the same result?

Kruskal's Algorithm: Correctness

It clearly generates a spanning tree. Call it T_K.

Suppose T_K is *not* minimum:

- Pick another spanning tree T_{min} with *lower cost* than T_K
- Pick the smallest edge $e_1=(u,v)$ in T_K that is **not** in T_{min}
- T_{min} already has a path p in T_{min} from u to v
  - ⇒ Adding e_1 to T_{min} will create a cycle in T_{min}
- Pick an edge e_2 in p that Kruskal's algorithm considered *after* adding e_1 (must exist: u and v unconnected when e_1 considered)
  - ⇒ cost(e_2) ≥ cost(e_1)
  - ⇒ can replace e_2 with e_1 in T_{min} without increasing cost!
- Keep doing this until T_{min} is identical to T_K
  - ⇒ T_K must also be minimal – contradiction!