Hashing

CSE 326
Data Structures
Lecture 15

Readings and References

- Reading
  - Chapter 5

Hashing

- Hashing is a family of data structures used to efficiently support insert, delete, find.
- It cannot be used efficently for other operations where the order of data is important. No list-all, range queries, successor, predecessor.

General Idea

- Key space of size M, but we only want to store subset of size N, where N<<M.
  - Keys are identifiers in programs. Compiler keeps track of them in a symbol table.
  - Keys are student names. We want to look up student records quickly by name.
  - Keys are chess configurations in a chess playing program.
  - Keys are URLs in a database of web pages.

Simple Hash Table

T

0	
1	
2	John Smith
3	
4	Judy Jones
5	
6	Martha Lee
7	Jerry Lee
8	
9	

Hash function:

h : U → { 0,1,…,Hsize -1}

U is the universe of keys

h("name") is the hash value of "name"

h(Judy Jones) = 4
h(Jerry Lee) = 7

Find("name") = T[h("name")]

Hashing Properties

- $\text{Load Factor} = \lambda = \dfrac{N}{\text{HSize}}$
  - Hash tables may have unused entries $\lambda < 1$
- Good quality hash function distribute data as evenly as possible over the keys.
- Collisions: h(inserted key) = h(existing key).
  - Open hashing - linked lists
  - Closed hashing - find a new place to put inserted key

Good Hash Functions

- Integers: Division method
  - Choose Hsize to be a prime
    h(n) = n mod Hsize
  - Example. Hsize = 23, h(50) = 4, h(1257) = 15
- Character Strings
  - $x = a_0a_1a_2 \ldots a_m$ is a character string. Define int(x) =
    $a_0 + a_1 128 + a_2 128^2 + \ldots + a_m 128^{m-1}$
    h(x) = int(x) mod Hsize
  - Compute h(x) using Horner's Rule

```
h :=0
for i = m to 0 by -1 do h := (a_i +128h) mod Hsize
return h
```


A Bad Hash Function

- Keys able1, able2, able3, able4
  - Hsize = 128
    int(ablex) mod 128 = int(a) = 97
    Thus, h(ablex) =h(abley) for all x and y
- Why use primes for hash table sizes?
  - Primes have no nontrivial divisors
  - Numbers relatively prime to 128 will also work for character strings

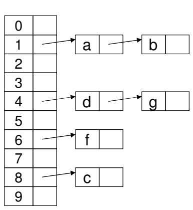
Multiplication Method

- Hash function defined by HSize and a floating point number A.
  - Integer case
  - $h(k) = \lfloor HSize * (k*A \bmod 1) \rfloor$
  - Example: HSize = 10, A = .485
    $h(50) = \lfloor 10 * (50*.485 \bmod 1) \rfloor$
    $= \lfloor 10*(24.25 \bmod 1) \rfloor$
    $= \lfloor 10*.25 \rfloor$
    $= 2$
    + HSize need not be prime
    - More computation than division method
- Another alternative – Universal Hashing

What about Collisions?

- Open Hashing - Collisions overflow into linked lists.
  - Load factors > 1 are possible
- Closed Hashing - if a collision occurs find another place in the hash table for the entry.
  - Load factor must be ≤ 1

Open Hashing (Chaining)



- h(a) = h(b) and h(d) = h(g)
- Chains may be ordered or unordered. Little advantage to ordering.

Open Hashing Properties

- Load factor = λ
  - Unsuccessful searches cost λ comparisons on average
  - Successful searches cost $1 + \lambda/2$ comparisons on average
- Comparisons can be expensive so choosing λ between 1/2 and 1 is wise.

Closed Hashing (Open Addressing)

- No chaining, every key fits in the hash table.
- Probe sequence
  - h(k)
  - (h(k) + f(1)) mod HSize
  - (h(k) + f(2)) mod HSize , …
- Insertion: Find the first probe with an empty slot.
- Find: Find the first probe that equals the query or is empty. Stop at HSize probe, in any case.
- Deletion: lazy deletion is needed. That is, mark locations as deleted, if a deleted key resides there.

Linear Probing

- f(i) = i
- Probe sequence
  - h(k)
  - (h(k) + 1) mod HSize
  - (h(k) + 2) mod HSize …
- Insertion (assuming $\lambda < 1$)

```
h := h(k)
while T(h) not empty do
  h := (h + 1) mod HSize;
insert k in T(h)
```


Linear Probing Example

	76	93	40	47	10	55
0				47	47	47
1						55
2		93	93	93	93	93
3					10	10
4						
5			40	40	40	40
6	76	76	76	76	76	76
Probes	1	1	1	3	1	3

Performance of Linear Probing

- If there is an available slot linear probing will find it.
- For large hash tables the expected number of probes on insertion is:

$$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)^2}\right)$$

- The expected number of probes on successful searches is:

$$\frac{1}{2}\left(1+\frac{1}{1-\lambda}\right)$$

- Linear probing suffers from primary clustering.
- Not a good idea to use linear probing with $\lambda > ½$.
- Lazy deletion needed.

Linear Probing – Clustering

[R. Sedgewick]

Quadratic Probing

- $f(i) = i^2$
- Probe sequence
  - h(k)
  - (h(k) + 1) mod HSize
  - (h(k) + 4) mod HSize
  - (h(k) + 9) mod HSize, …
- Insertion (assuming $\lambda < 1/2$)

```
h := h(k);
i := 0;
while T(h) not empty do {
  h := (h + 2*i + 1) mod HSize;
  i := i + 1 }
insert k in T(h)
```

Note: $(i+1)^2 - i^2 = 2i + 1$

Quadratic Probing Works for $\lambda < 1/2$

- If HSize is prime then $(h(x) + i^2) \bmod HSize \neq (h(x) + j^2) \bmod HSize$ for $i \neq j$ and $0 \le i,j < HSize/2$.
- Proof

  $(h(x) + i^2) \bmod HSize = (h(x) + j^2) \bmod HSize$

  $(h(x) + i^2) - (h(x) + j^2) \bmod HSize = 0$

  $(i^2 - j^2) \bmod HSize = 0$

  $(i-j)(i+j) \bmod HSize = 0$

  $\Rightarrow\Leftarrow$ HSize does not divide (i-j) or (i+j)

Quadratic Probing may Fail if $\lambda \ge 1/2$

51

0	
1	
2	16
3	45
4	59
5	
6	76

51 mod 7 = 2 ; i = 0
(2 + 1) mod 7 = 3; i = 1
(3 + 3) mod 7 = 6; i = 2
(6 + 5) mod 7 = 4; i = 3
(4 + 7) mod 7 = 4; i = 4
(4 + 9) mod 7 = 6; i = 5
(6 + 11) mod 7 = 3; i = 6
(3 + 13) mod 7 = 2, i = 7
…

Performance of Quadratic Probing

- Although quadratic probing can fail for $\lambda \ge$ ½, it is not likely to do so. We can use load factors greater than ½, but load factors close to 1 should be avoided.
- Quadratic hashing does not suffer from primary clustering, but has only minor secondary clustering.
- With load factors near ½ the expected number of probes per successful search is about 1.5.
- Lazy deletion must be used.

Double Hashing

- f(i) = i g(k) where g is a second hash function
- Probe sequence

  h(k)
  (h(k) + g(k)) mod HSize
  (h(k) + 2g(k)) mod HSize
  (h(k) + 3g(k)) mod HSize, …
- In choosing g care must be taken so that it never evaluates to 0.
- A good choice for gis to choose a prime R < HSize and let g(k) = R – (k mod R).

Double Hashing Example

h(k) = k mod 7 and g(k) = 5 – (k mod 5)

	76	93	40	47	10	55
0						
1				47	47	47
2		93	93	93	93	93
3					10	10
4						55
5			40	40	40	40
6	76	76	76	76	76	76
Probes	1	1	1	2	1	2

Double Hashing is Safe for $\lambda < 1$

- Let h(k) = k mod p and g(k) = q – (k mod q) where 2 < q < p and p and q are primes. The probe sequence h(k) + ig(k) mod p probes every entry of the hash table.

  Let $0 \le m < p$, h = h(k), and g = g(k). We show that h+ig mod p = m for some i. 0 < g < p, so g and p are relatively prime. By extended Euclid's algorithm that are s and t such that

  sg + tp = 1. Choose i = (m-h)s mod p
  (h + ig) mod p =
  (h + (m-h)sg) mod p =
  (h + (m-h)sg + (m-h)tp) mod p =
  (h + (m-h)(sg + tp) mod p =
  (h + (m-h)) mod p = m mod p = m

Deletion in Hashing

- Open hashing (chaining) – no problem
- Closed hashing – must do lazy deletion. Deleted keys are marked as deleted.
  - Find: done normally
  - Insert: treat marked slot as an empty slot and fill it.

h(k) = k mod 7
Linear probing

Find 59

0	
1	
2	16
3	23
4	59
5	
6	76

Insert 30

0	
1	
2	16
3	30
4	59
5	
6	76

Rehashing

- Build a bigger hash table of approximately twice the size when λ exceeds a particular value
  - Go through old hash table, ignoring items marked deleted
  - Recompute hash value for each non-deleted key and put the item in new position in new table
  - Cannot just copy data from old table because the bigger table has a new hash function
- Running time is O(N) but happens very infrequently
  - Not good for real-time safety critical applications

Rehashing Example

- Open hashing – $h_1(x) = x \bmod 5$ rehashes to $h_2(x) = x \bmod 11$.

$\lambda = 1$

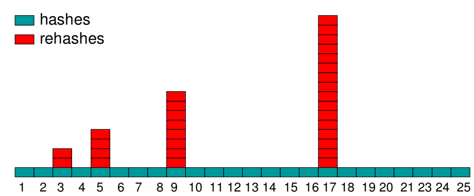
0	1	2	3	4
25		37	83	
		52	98	

$\lambda = 5/11$

0	1	2	3	4	5	6	7	8	9	10
			25	37		83		52		98

Rehashing Picture

- Starting with table of size 2, double when load factor > 1.



Amortized Analysis of Rehashing

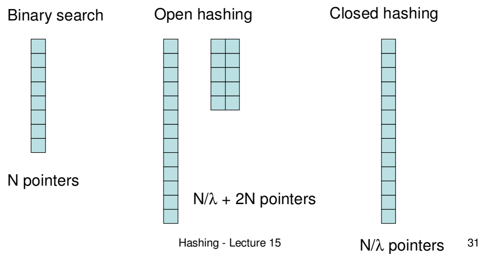
- Cost of inserting n keys is < 3n
- $2^k + 1 \leq n \leq 2^{k+1}$
  - Hashes = n
  - Rehashes = $2 + 2^2 + \ldots + 2^k = 2^{k+1} - 2$
  - Total = $n + 2^{k+1} - 2 < 3n$
- Example
  - n = 33, Total = 33 + 64 –2 = 95 < 99

Case Study

- Spelling Dictionary - 30,000 words
- Goals
  - Fast spell checking
  - Minimal storage
- Possible solutions
  - Sorted array and binary search
  - Open hashing (chaining)
  - Closed hashing with linear probing
- Notes
  - Almost all searches are successful
  - 30,000 word average 8 bytes per word, 240,000 bytes
  - Pointers are 4 bytes

Storage

- Assume word are stored as strings and entries in the arrays are pointers to the strings.

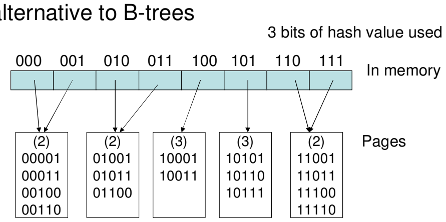


Analysis

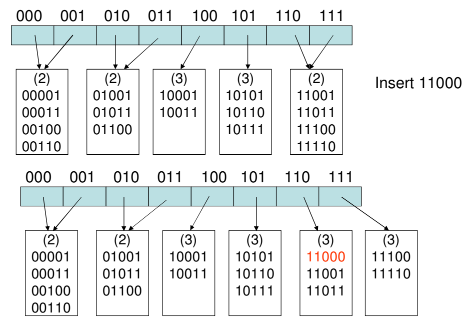
- Binary Search
  - Storage = N pointers + words = 360,000 bytes
  - Time = $\log_2 N \leq 15$ probes in worst case
- Open hashing
  - Storage = 2N + N/ λ pointers + words
    λ = 1 implies 600,000 bytes
  - Time = 1 + λ/2 probes per access
    λ = 1 implies 1.5 probes per access
- Closed hashing
  - Storage = N/ λ pointers + words
    λ = 1/2 implies 480,000 bytes
  - Time = (1/2)(1+1/(1-λ)) probes
    λ = 1/2 implies 1.5 probes per access

Extendible Hashing

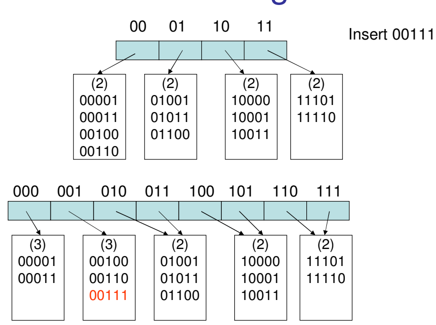
- Extendible hashing is a technique for storing large data sets that do not fit in memory.
- An alternative to B-trees



Splitting



Rehashing



Analysis of Extendible Hashing

- On deletion neighbors can be merged.
- If table uses k bits but all pages use k-1 bits then rehashing to a smaller table can be done. Not normally an issue with large databases.
- Rehashing does not touch pages.
- Splitting and merging touch only two pages.

Fingerprints

- Given a string x we want a fingerprint x' with the properties.
  - x' is short, say 128 bits
  - Given $x \neq y$ the probability that $x' = y'$ is infintesimal (almost zero)
  - Computing x' is very fast
- MD5 - Message Digest Algorithm 5 is a recognized standard
- Applications in databases and cryptography

Fingerprint Math

Given 128 bits and N strings what is the probability that the fingerprints of two strings coincide?

$$1-\frac{2^{128}(2^{128}-1)\cdots(2^{128}-N+1)}{(2^{128})^N}$$

This is essentially zero for $N < 2^{40}$.

Hashing Summary

- Hashing is one of the most important data structures.
- Hashing has many applications where operations are limited to find, insert, and delete.
- Dynamic hash tables have good amortized complexity.
- Extendible hashing is useful in databases.
- Fingerprints good for databases and crypto.