CSE 331
Software Design & Implementation

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Lecture 3 – Writing Loops

Checking Correctness of a Loop

Consider a while-loop (other loop forms not too different) with a loop invariant **I**.

```
{{ P }}
 S1

{{ Inv: I }}
 while (cond)
   S2

 S3
{{ Q }}
```

Informally, we need:

- **I** holds initially
- **I** holds each time around
- **Q** holds after we exit

Formally, we need validity of:

- {{ P }} S1 {{ **I** }}
- {{ **I** and cond }} S2 {{ **I** }}
- {{ **I** and not cond }} S3 {{ Q }}

(can check these with backward reasoning instead)

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
i = 0;
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
{{ s = b[0] + ... + b[n-1] }}
```

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
```

{{ s = 0 and i = 0 }}

{{ Inv: s = b[0] + ... + b[i-1] }}

```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

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s = 0;
i = 0;
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{{ s = 0 and i = 0 }}

{{ Inv: s = b[0] + ... + b[i-1] }}

```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies
  s = b[0] + … + b[i-1] ?

Yes. (An empty sum is zero.)

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
```

{{ s = 0 and i = 0 }}

{{ Inv: s = b[0] + ... + b[i-1] }}

```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies
  s = b[0] + … + b[i-1] ?

More formal

s = sum of all b[k] with $0 \leq k \leq i-1$

i = 3 ($0 \leq k \leq 2$): s = b[0] + b[1] + b[2]
i = 2 ($0 \leq k \leq 1$): s = b[0] + b[1]
i = 1 ($0 \leq k \leq 0$): s = b[0]
i = 0 ($0 \leq k \leq -1$) s = 0

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = 0;
```
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```
{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies **I**

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
i = 0;
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  {{ s = b[0] + ... + b[i-1] and i != n }}
  s = s + b[i];
  i = i + 1;
  {{ s = b[0] + ... + b[i-1] }}
}
{{ s = b[0] + ... + b[n-1] }}
```

- (s = 0 and i = 0) implies **`I`**
- {{ **`I`** and i != n }} `S` {{ **`I`** }} ?

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = 0;
```
{{ Inv: s = b[0] + ... + b[i-1] }}
```
while (i != n) {
```
  {{ s = b[0] + ... + b[i-1] and i != n }}
```
  s = s + b[i];
  i = i + 1;
```
  {{ s = b[0] + ... + b[i-1] }}
```
}
```
{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies **I**
- {{ **I** and i != n }} S {{ **I** }} ?

{{ s + b[i] = b[0] + … + b[i] }}
{{ s = b[0] + … + b[i] }}

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
```

{{ Inv: s = b[0] + ... + b[i-1] }}

```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[i-1] and not (i != n) }}

{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies **`I`**
- {{ **`I`** and i != n }} `S` {{ **`I`** }}
- {{ **`I`** and not (i != n) }} implies s = b[0] + … + b[n-1] ?

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
i = 0;
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
{{ s = b[0] + ... + b[n-1] }}
```

- (s = 0 and i = 0) implies **`I`**
- {{ **`I`** and i != n }} `S` {{ **`I`** }}
- {{ **`I`** and i = n }} implies `Q`

These three checks verify that the outermost triple is valid (i.e., that the code is correct).

Termination

- Technically, this analysis does not check that the code **terminates**
  - it shows that the postcondition holds if the loop exits
  - but we never showed that the loop actually exits
- However, that follows from an analysis of the running time
  - e.g., if the code runs in $O(n^2)$ time, then it terminates
  - an infinite loop would be O(infinity)
  - any finite bound on the running time proves it terminates
- It is normal to also analyze the running time of code we write, so we get termination already from that analysis.

Example: sum of array (attempt 2)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = -1;
```

{{ Inv: s = b[0] + ... + b[i] }}   Changed

```
while (i != n-1) {
  i = i + 1;
  s = s + b[i];
}
```

{{ s = b[0] + ... + b[n-1] }}

Example: sum of array (attempt 2)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = -1;
```

Changed from i = 0

{{ Inv: s = b[0] + ... + b[i] }}

```
while (i != n-1) {
  i = i + 1;
  s = s + b[i];
}
```

Changed from n

Reordered

{{ s = b[0] + ... + b[n-1] }}

Example: sum of array (attempt 2)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = -1;
```

{{ Inv: s = b[0] + ... + b[i] }}

```
while (i != n-1) {
  i = i + 1;
  s = s + b[i];
}
```

{{ s = b[0] + ... + b[n-1] }}

Work as before:

- (s = 0 and i = -1) implies **I**
  - **I** holds initially
- (**I** and i = n-1) implies **Q**
  - **I** implies **Q** at exit

Example: sum of array (attempt 2)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = -1;
```

{{ Inv: s = b[0] + ... + b[i] }}

```
while (i != n-1) {
  i = i + 1;
  s = s + b[i];
}
```

{{ s = b[0] + ... + b[n-1] }}

{{ s + b[i+1] = b[0] + ... + b[i+1] }}

{{ s + b[i] = b[0] + … + b[i] }}

{{ s = b[0] + ... + b[i] }}

Example: sum of array (attempt 2)

Consider the following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
i = -1;
{{ Inv: s = b[0] + ... + b[i] }}
while (i != n-1) {
  i = i + 1;
  s = s + b[i];
}
{{ s = b[0] + ... + b[n-1] }}
```

- (s = 0 and i = -1) implies **`I`**
  - as before
- {{ **`I`** and i != n-1 }} `S` {{ **`I`** }}
  - reason backward
- (**`I`** and i = n-1) implies **`Q`**
  - as before

Example: sum of array (attempt 3)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = -1;
```

{{ Inv: s = b[0] + ... + b[i] }}

```
while (i != n-1) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

Suppose we miss-order the assignments to `i` and `s`…

Where does the correctness check fail?

Example: sum of array (attempt 3)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = -1;
```
{{ Inv: s = b[0] + ... + b[i] }}
```
while (i != n-1) {
  s = s + b[i];
  i = i + 1;
}
```
{{ s = b[0] + ... + b[n-1] }}

Suppose we miss-order the assignments to `i` and `s`…

We can spot this bug because the invariant does not hold:

{{ s + b[i] = b[0] + ... + b[i+1] }}
{{ s = b[0] + ... + b[i+1] }}
{{ s = b[0] + ... + b[i] }}

First assertion is not Inv.

Example: sum of array (attempt 3)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = -1;
```

{{ Inv: s = b[0] + ... + b[i] }}

```
while (i != n-1) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

Suppose we miss-order the assignments to `i` and `s`…

We can spot this bug because the invariant does not hold:

{{ s = b[0] + ... + b[i-1] + b[i+1] }}

For example, if i = 2, then

s = b[0] + b[1] + b[2] vs
s = b[0] + b[1] + b[3]

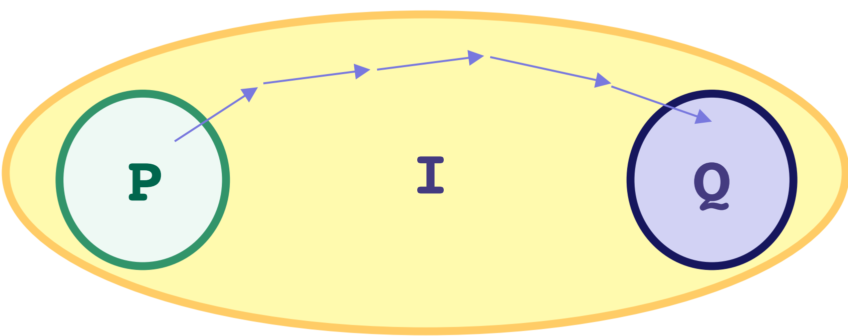
Thinking About Loop Invariants

{{ **P** }} `while (cond) S` {{ **Q** }}

This triple is valid iff

{{ **P** }}
{{ Inv: **I** }}
```
while (cond)
  S
```
{{ **Q** }}

- **I** holds initially
- **I** holds each time we execute `S`
- **Q** holds when **I** holds and `cond` is false



Thinking About Loop Invariants

- Loop invariant comes out of the algorithm idea
  - describes partial progress toward the goal
  - how you will get from start to end

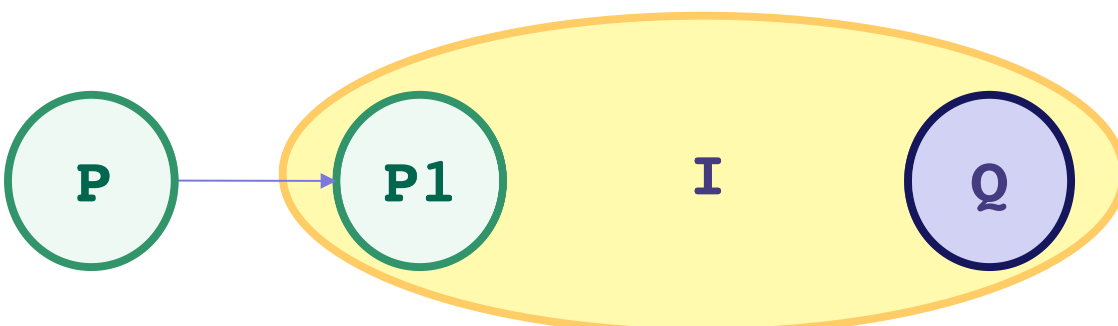
- Essence of the algorithm idea is:
  - invariant
  - how you make progress on each step (e.g., `i = i + 1`)

- Code is *ideally* just details...

Loop Invariant → Code

In fact, can usually deduce the code from the invariant:

- When does loop invariant satisfy the postcondition?
  - gives you the termination condition
- What is the easiest way to satisfy the loop invariant?
  - gives you the initialization code
- How does the invariant change as you make progress?
  - gives you the rest of the loop body



Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
??
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (?) {

   ??

}
```

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
??
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (?) {

   ??

}
```

{{ m = max(b[0], ..., b[n-1]) }}

When does Inv imply postcondition?

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
??
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (?) {

  ??

}
```

{{ m = max(b[0], ..., b[n-1]) }}

When does Inv imply postcondition?
Happens when i = n

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
??
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {

    ??

}
```

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
??
```

Easiest way to make this hold?

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {

   ??

}
```

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
??
```

Easiest way to make this hold?
Take i = 1 and m = max(b[0])

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {

   ??

}
```

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {

   ??

}
```

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {

   ??

}
```

{{ m = max(b[0], ..., b[n-1]) }}

How do we progress toward termination?
(comes from the algorithm idea)

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {

   ??
   i = i + 1;
}
```

{{ m = max(b[0], ..., b[n-1]) }}

How do we progress toward termination? We start at i = 1 and end at i = n, so Try this.

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {

   ??
   i = i + 1;
}
```

↑ {{ m = max(b[0], …, b[i]) }}

{{ m = max(b[0], …, b[i-1]) }}

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

Set m = max(m, b[i])

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n)  {
```

{{ m = max(b[0], …, b[i-1]) }}

```
   ??
```

{{ m = max(b[0], …, b[i]) }}

```
   i = i + 1;
```

{{ m = max(b[0], …, b[i-1]) }}

```
}
```

How do we fill this in?

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

Set m = max(m, b[i])

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {
  if (b[i] > m)          OR m = Math.max(m, b[i]);
    m = b[i];
  i = i + 1;
}
```

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {
  if (b[i] > m)
    m = b[i];
  i = i + 1;
}
```

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {
  if (b[i] > m)
    m = b[i];
  i = i + 1;
}
```

{{ m = max(b[0], ..., b[n-1]) }}

the algorithm idea

Invariants are Essential

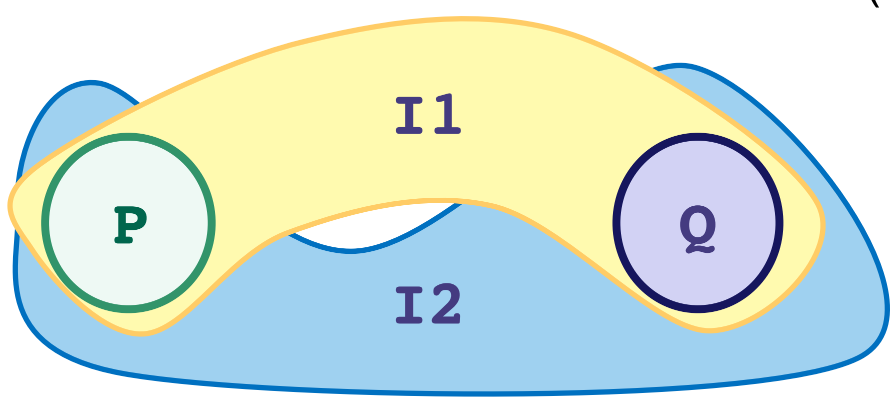
Invariant + progress step is the essence of the algorithm idea

- rest is hopefully just details that follow from the invariant

Work toward thinking at the level of invariants not code

- gain confidence that you can do the rest without difficulty

m = max(b[0], ..., b[i-1])



m = max(b[i], ..., b[n-1])

Loop Invariant Design Pattern

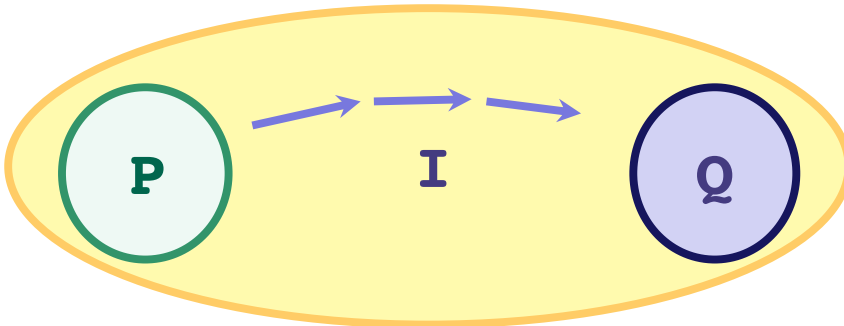
Loop invariant is often a weakening of the postcondition

- partial progress with completion a special case
- small enough weakening that Inv + one condition gives Q

1. sum of array
   - postcondition: s = b[0] + b[1] + … + b[n-1]
   - loop invariant: s = b[0] + b[1] + … + b[i-1]
     - gives postcondition when i = n
2. max of array
   - postcondition: m = max(b[0], b[1], …, b[n-1])
   - loop invariant: m = max(b[0], b[1], ..., b[i-1])
     - gives postcondition when i = n

Loop Invariant Design Patterns

Algorithm Idea formalized in: Invariant + *progress step*



- how do you make progress toward termination?
  - if condition is i != n (and i <= n)
    try i = i + 1
  - if condition is i != j (and i <= j)
    try i = i + 1 or j = j – 1

Finding the loop invariant

Not every loop invariant is simple weakening of postcondition, but…

- that is the easiest case
- it happens a lot

In this class (e.g., homework):

- if I ask you to find the invariant, it will *very likely* be of this type
  - I will ask you to write more complex code when the invariant given
  - I will you to check correctness of even more complex code
  - HW2-4 will practice these
- to learn about more ways of finding invariants: CSE 421