CSE 332 Autumn 2023
Lecture 26: Topological Sort and Minimum Spanning Trees

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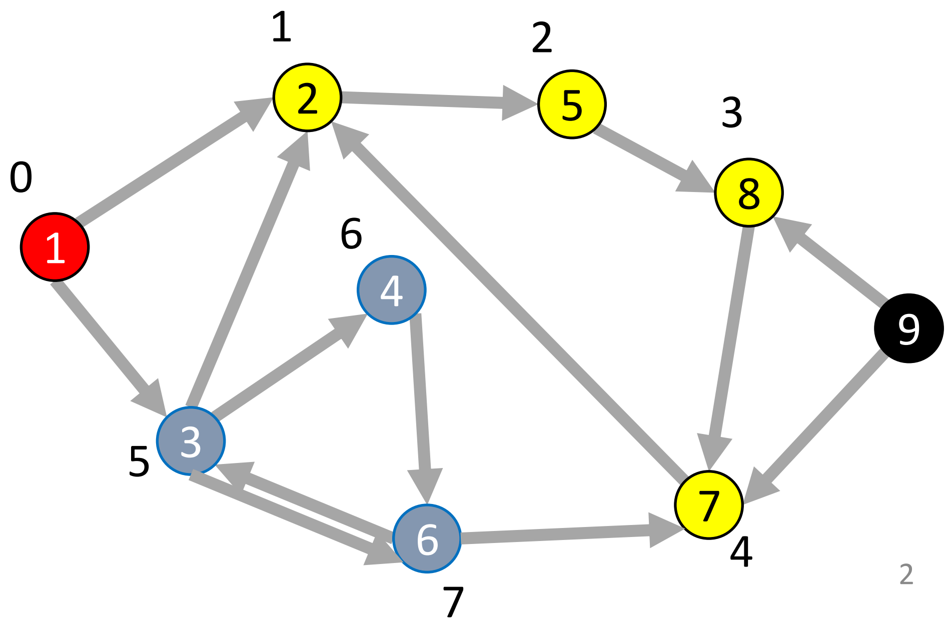
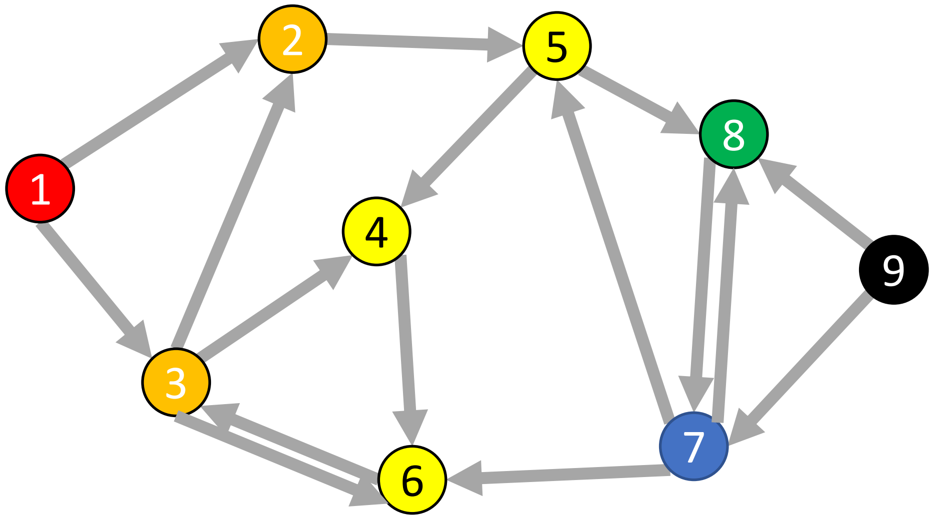
http://www.cs.uw.edu/332

Depth-First Search

- Input: a node s
- Behavior: Start with node s, visit one neighbor of s, then all nodes reachable from that neighbor of s, then another neighbor of s,…
- Output:
  - Does the graph have a cycle?
  - A **topological sort** of the graph.



DFS (non-recursive)

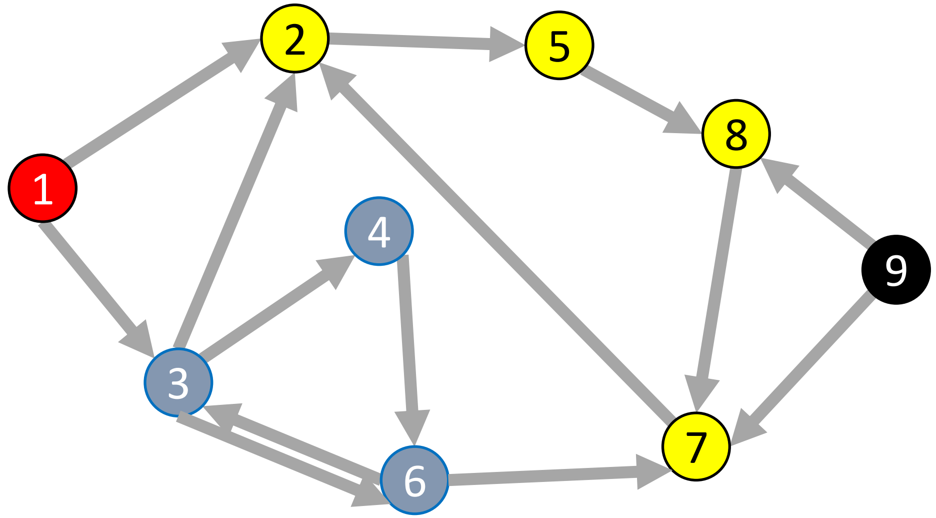


Running time: $\Theta(|V| + |E|)$

```
void dfs(graph, s){
        found = new Stack();
        found.pop(s);
        mark s as "visited";
        While (!found.isEmpty()){
                current = found.pop();
                for (v : neighbors(current)){
                        if (! v marked "visited"){
                                mark v as "visited";
                                found.push(v);
                        }
                }
        }
}
```

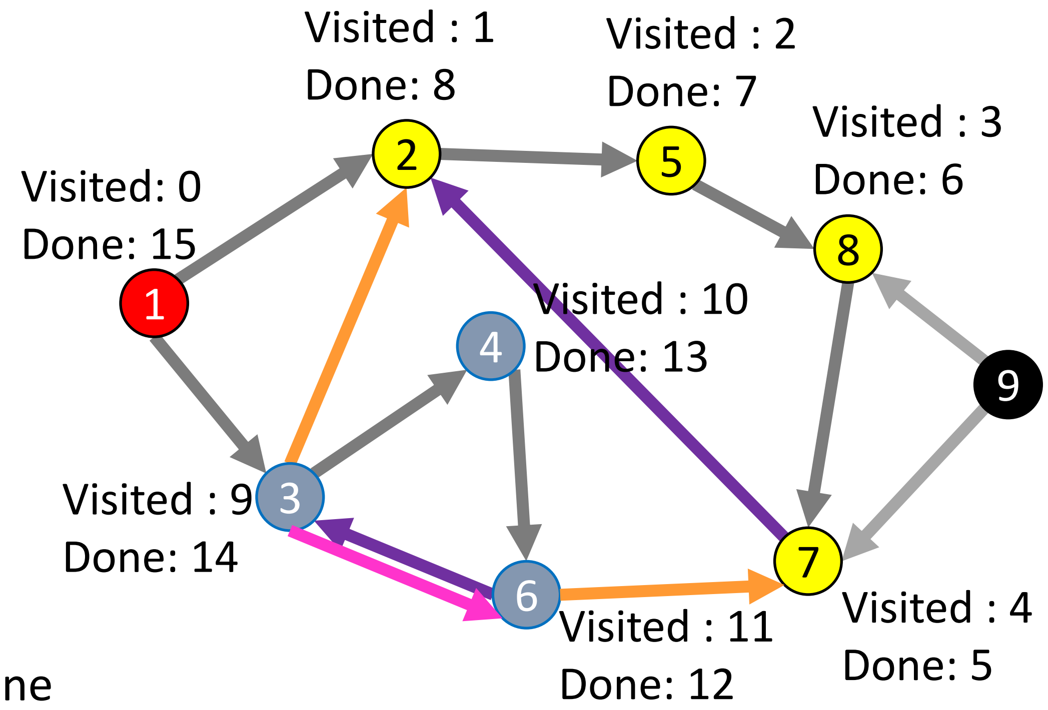
DFS Recursively (more common)

```
void dfs(graph, curr){
        mark curr as "visited";
        for (v : neighbors(current)){
                if (! v marked "visited"){
                        dfs(graph, v);
                }
        }
        mark curr as "done";
}
```



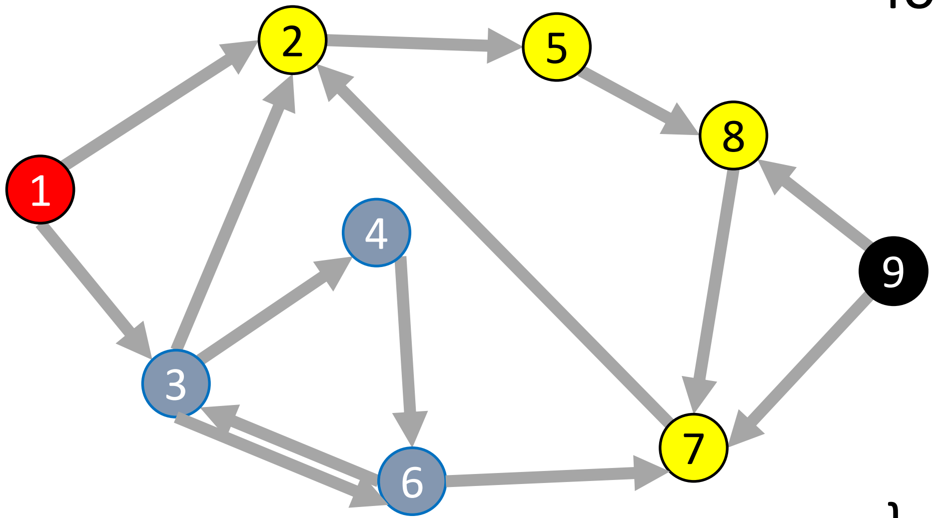
Using DFS

- Consider the "visited times" and "done times"
- Edges can be categorized:
  - Tree Edge
    - (a, b) was followed when pushing
    - (a, b) when b was unvisited when we were at a
  - Back Edge
    - (a, b) goes to an "ancestor"
    - a and b visited but not done when we saw (a, b)
    - $t_{visited}(b) < t_{visited}(a) < t_{done}(a) < t_{done}(b)$
  - Forward Edge
    - (a, b) goes to a "descendent"
    - b was visited and done between when a was visited and done
    - $t_{visited}(a) < t_{visited}(b) < t_{done}(b) < t_{done}(a)$
  - Cross Edge
    - (a, b) goes to a node that doesn't connect to a
    - b was seen and done before a was ever visited
    - $t_{done}(b) < t_{visited}(a)$



Cycle Detection

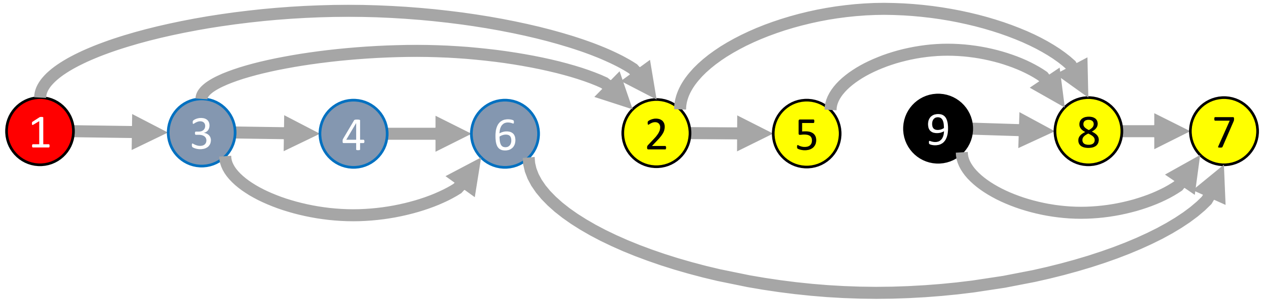
Idea: Look for a back edge!

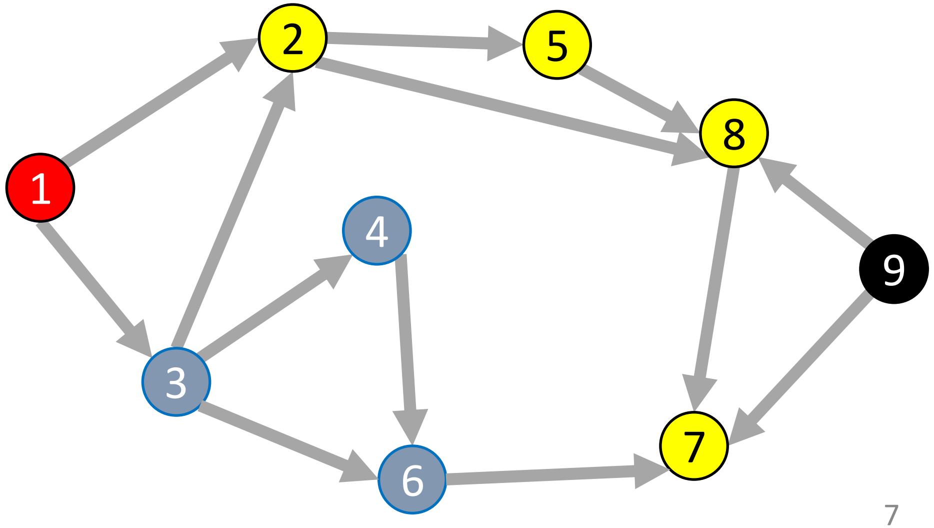



```
boolean hasCycle(graph, curr){
        mark curr as "visited";
        cycleFound = false;
        for (v : neighbors(current)){
                if (v marked "visited" && ! v marked "done"){
                        cycleFound=true;
                }
                if (! v marked "visited" && !cycleFound){
                        cycleFound = hasCycle(graph, v);
                }
        }
        mark curr as "done";
        return cycleFound;
}
```

Topological Sort

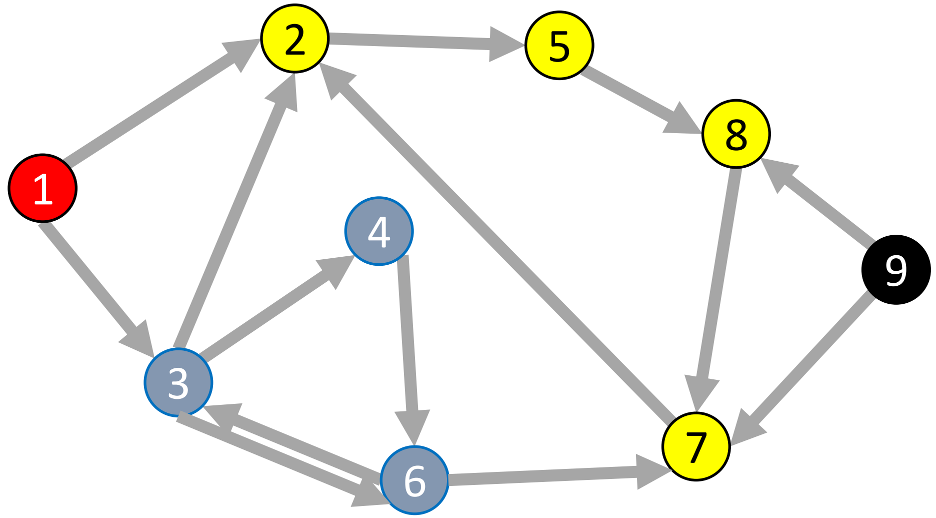
- A Topological Sort of a **directed acyclic graph** $\boldsymbol{G = (V, E)}$ is a permutation of V such that if $(u, v) \in E$ then u is before v in the permutation





DFS Recursively

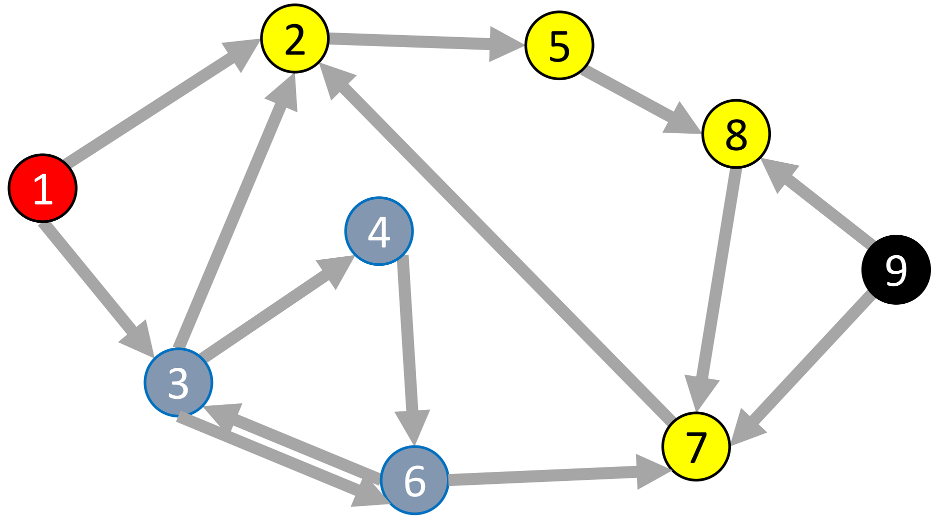
```
void dfs(graph, curr){
        mark curr as "visited";
        for (v : neighbors(current)){
                if (! v marked "visited"){
                        dfs(graph, v);
                }
        }
        mark curr as "done";
}
```



DFS Recursively

```
void dfs(graph, curr){
        mark curr as “visited”;
        for (v : neighbors(current)){
                if (! v marked “visited”){
                        dfs(graph, v);
                }
        }
        mark curr as “done”;
}
```

Idea: List in reverse order by “done” time



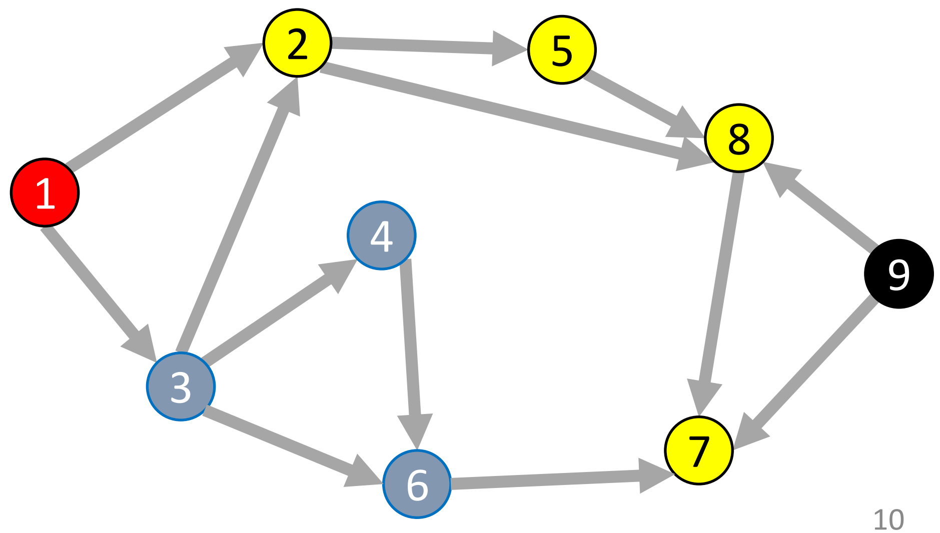
DFS: Topological sort

```
List topSort(graph){
        List<Nodes> done = new List<>();
        for (Node v : graph.vertices){
                if (!v.visited){
                        finishTime(graph, v, finished);
                }
        }
        done.reverse();
        return done;
}

void finishTime(graph, curr, finished){
        curr.visited = true;
        for (Node v : curr.neighbors){
                if (!v.visited){
                        finishTime(graph, v, finished);
                }
        }
        done.add(curr)
}
```

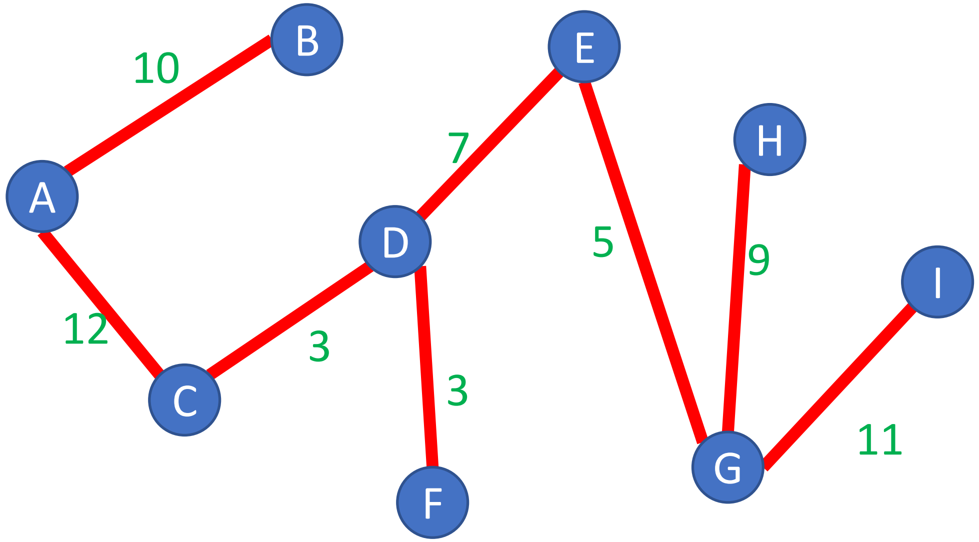
Idea: List in reverse order by "done" time

finished:



Definition: Tree

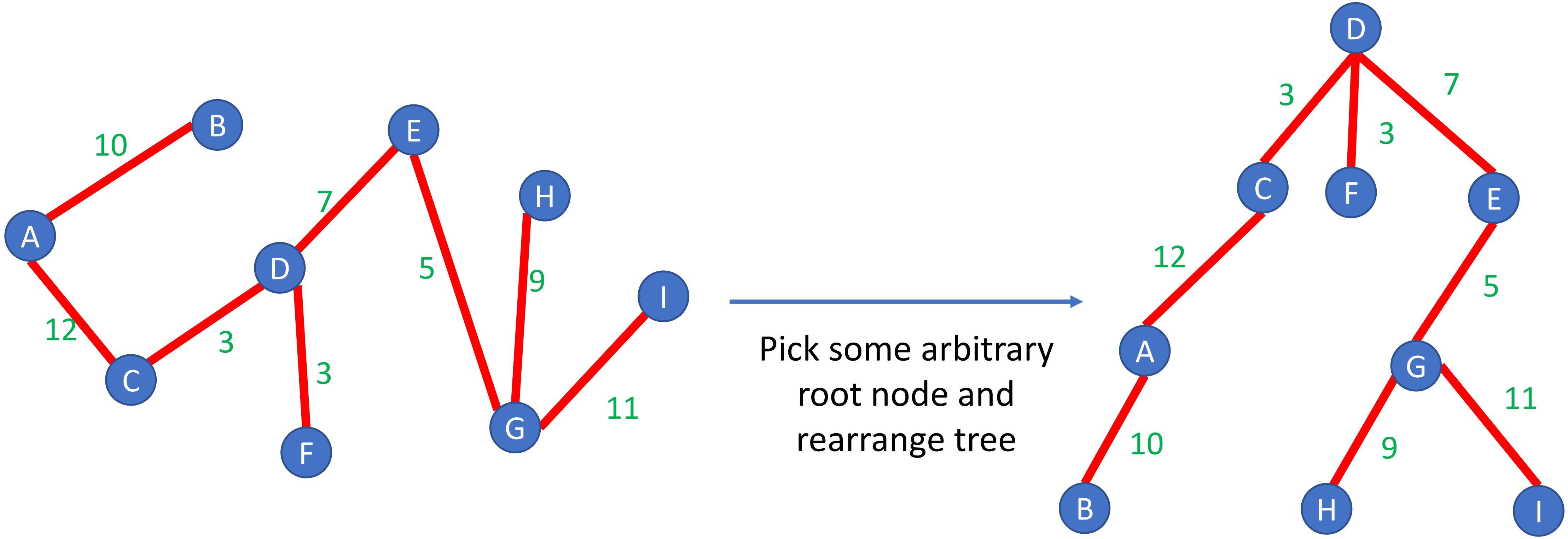
A connected graph with no cycles



Note: A tree does not need a root, but they often do!

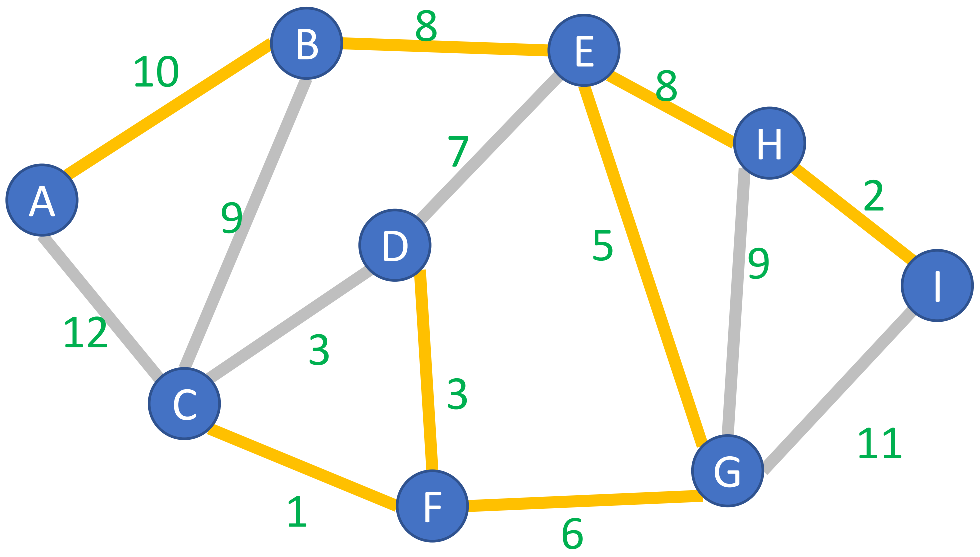
Definition: Tree

A connected graph with no cycles



Definition: Spanning Tree

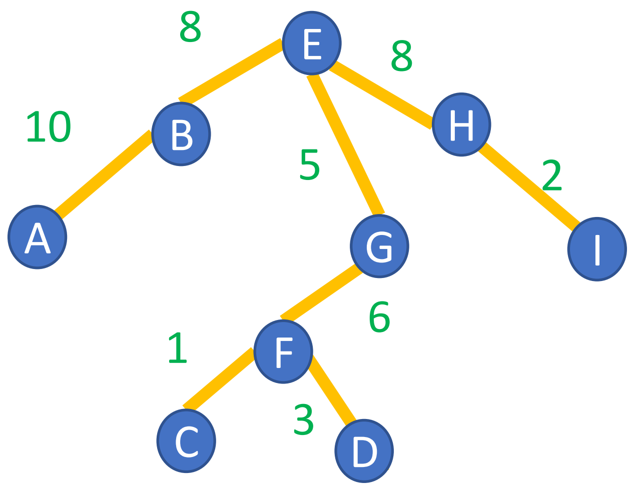
A Tree $\boldsymbol{T} = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$



Pick some arbitrary root node and rearrange tree

How many edges does T have?

$V - 1$

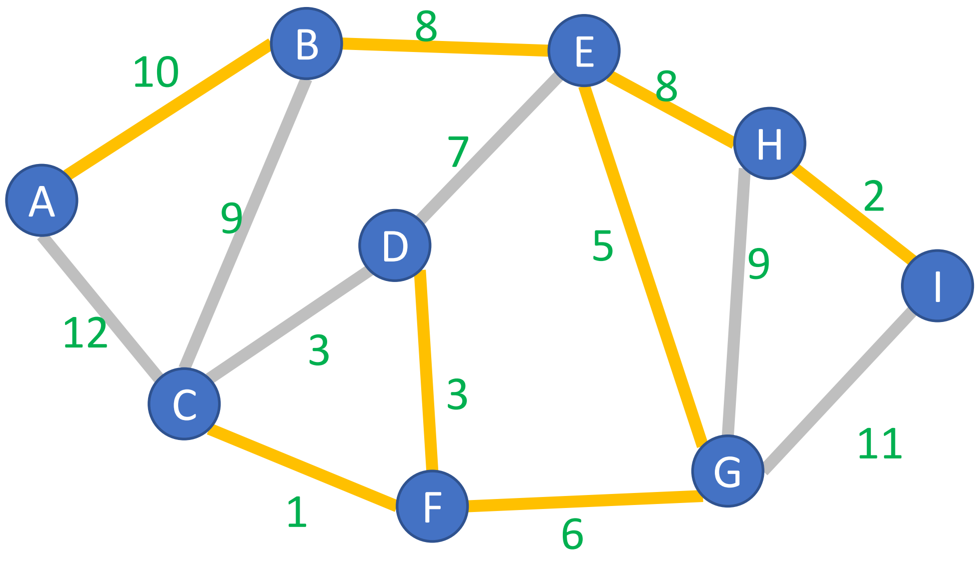


Any set of V-1 edges in the graph that doesn't have any cycles is guaranteed to be a spanning tree!

Any set of V-1 edges that connects all the nodes in the graph is guaranteed to be a spanning tree!

Definition: Minimum Spanning Tree

A Tree $\boldsymbol{T} = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$, that has minimal cost

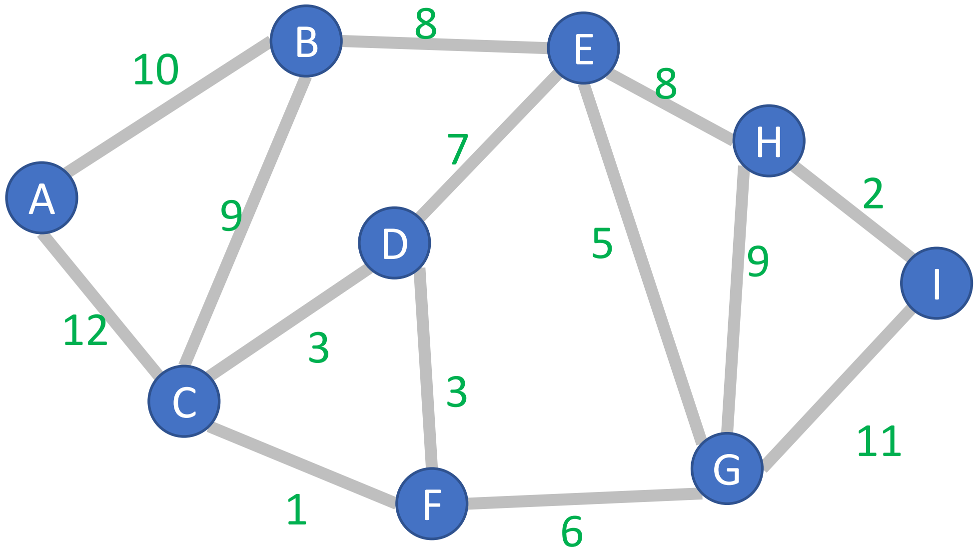


$$Cost(T) = \sum_{e \in E_T} w(e)$$

Kruskal's Algorithm

Start with an empty tree A

Add to A the lowest-weight edge that does not create a cycle



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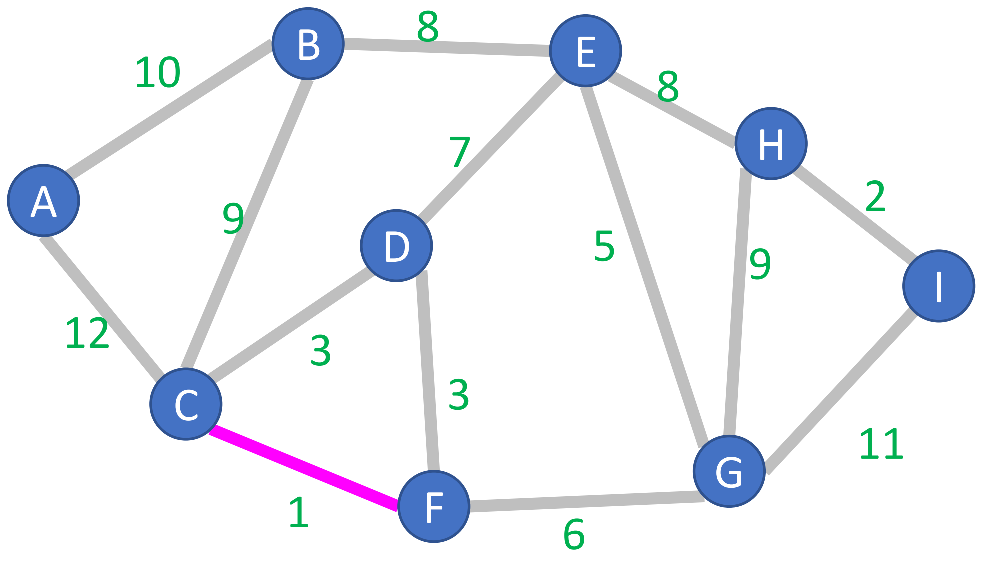

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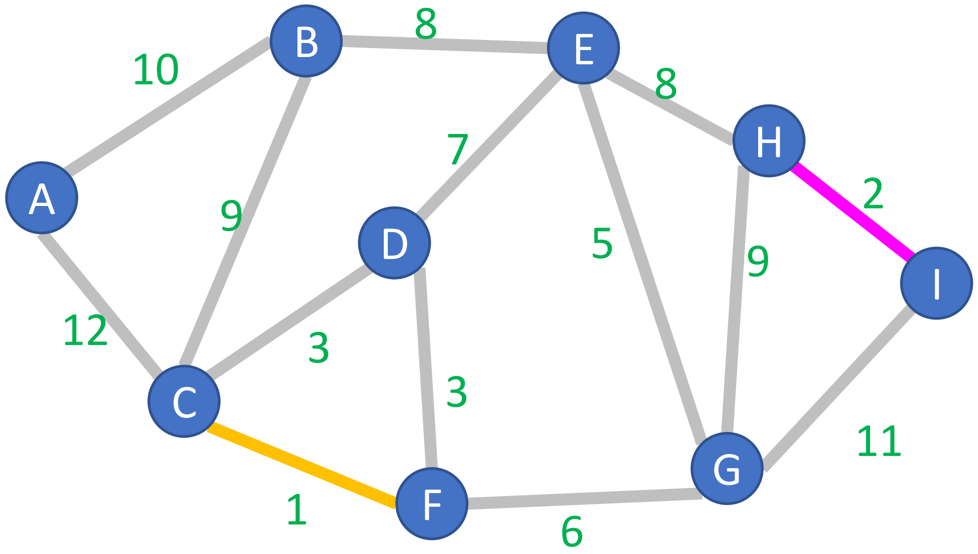

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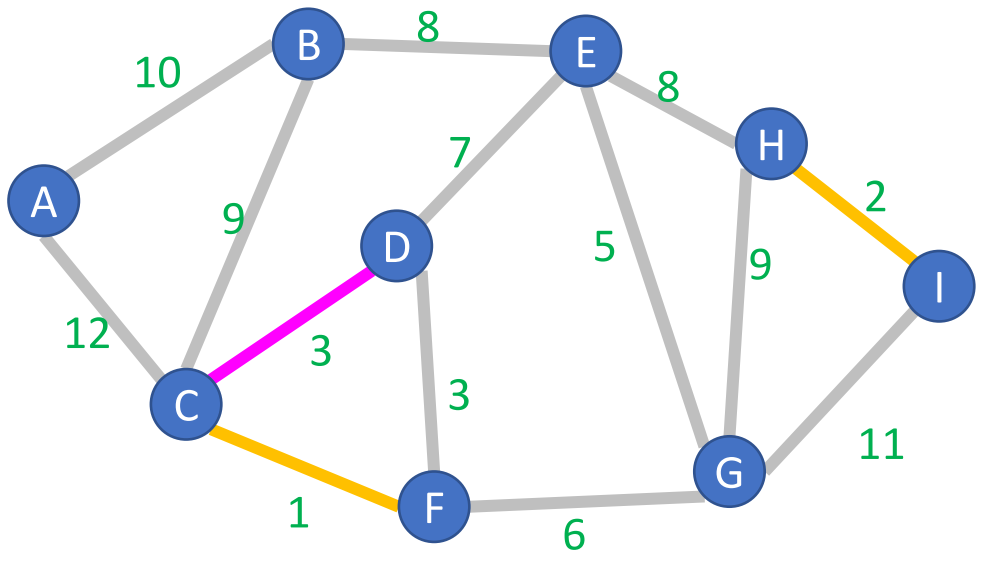

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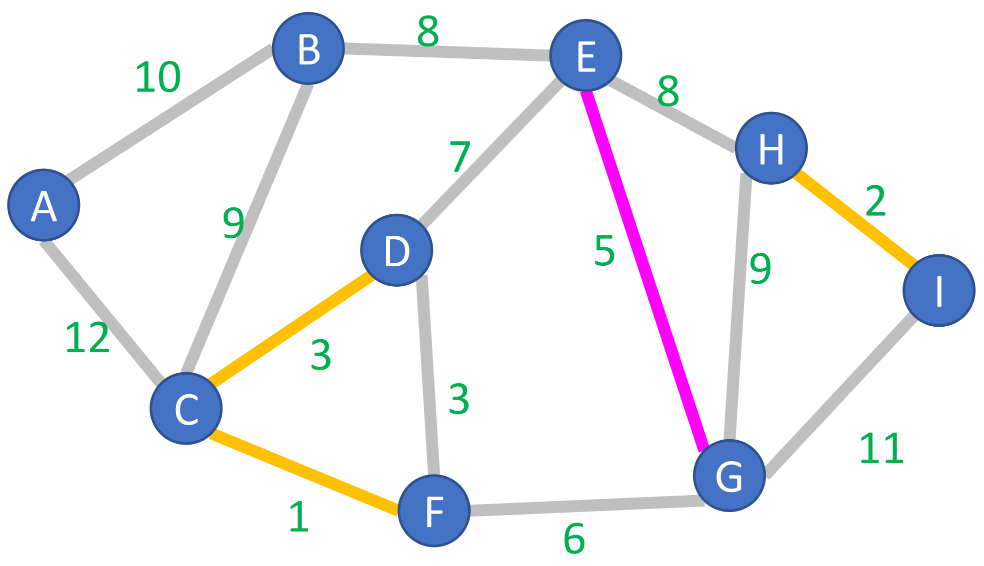

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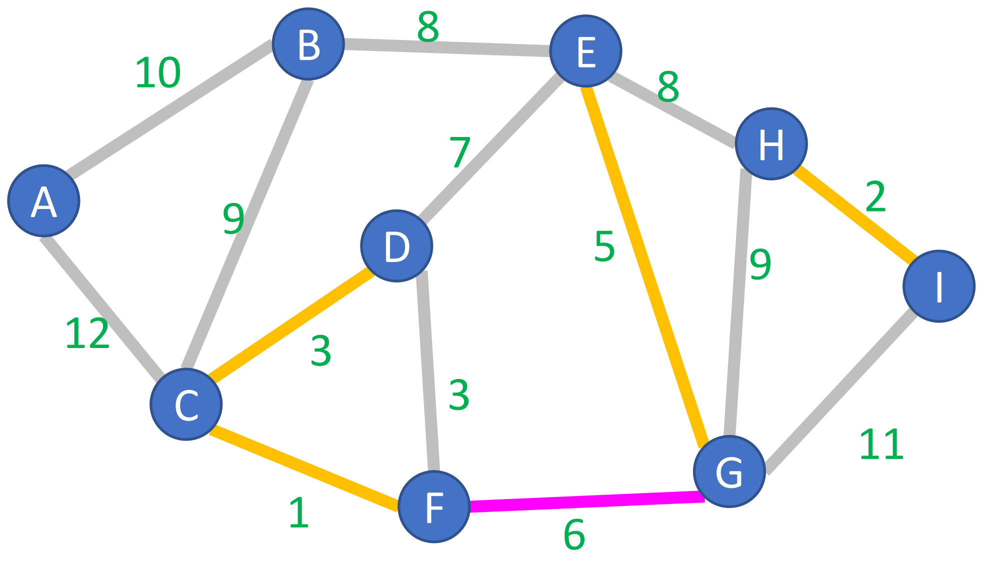
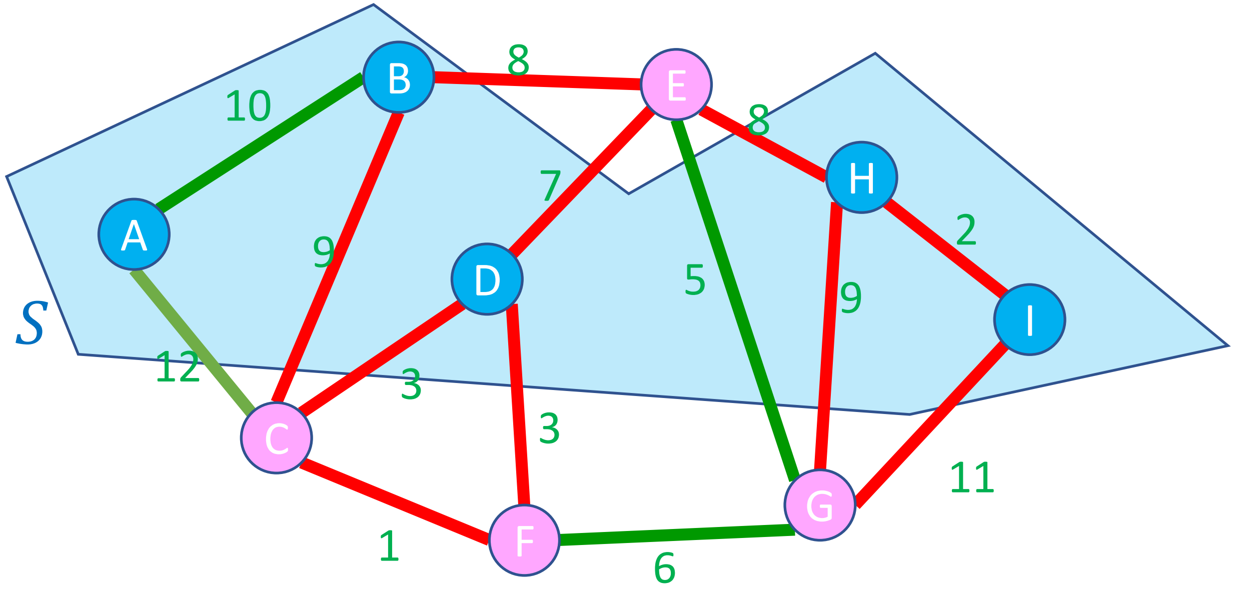


Definition: Cut

A Cut of graph $G = (V, E)$ is a partition of the nodes into two sets, S and $V - S$



Edge $(v_1, v_2) \in E$ crosses a cut if $v_1 \in S$ and $v_2 \in V - S$ (or opposite), e.g. (A, C)

A set of edges R Respects a cut if no edges cross the cut e.g. $R = \{(A,B),(E,G),(F,G)\}$

Cut Theorem

If a set of edges A is a subset of a minimum spanning tree T, let $(S, V - S)$ be any cut which A respects. Let e be the least-weight edge which crosses $(S, V - S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.

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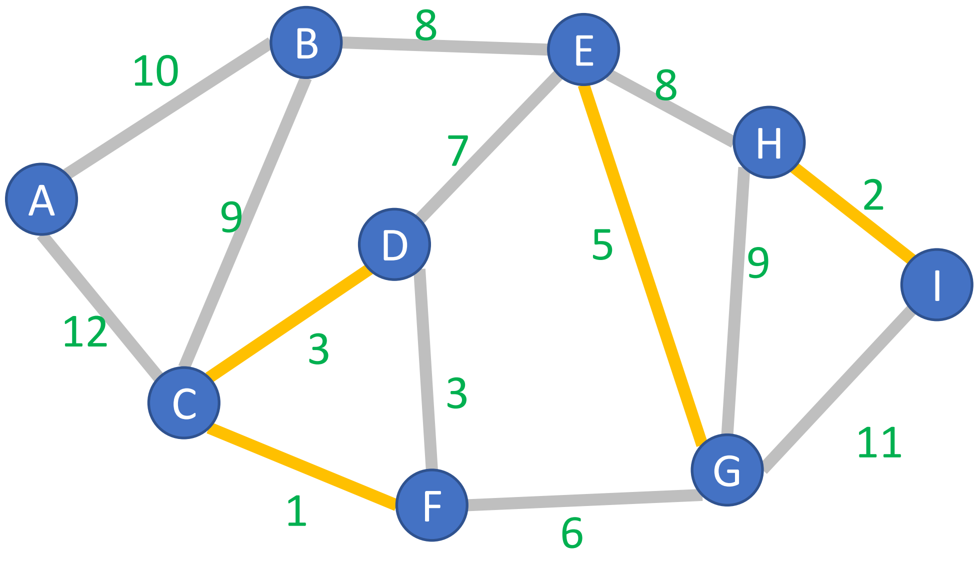

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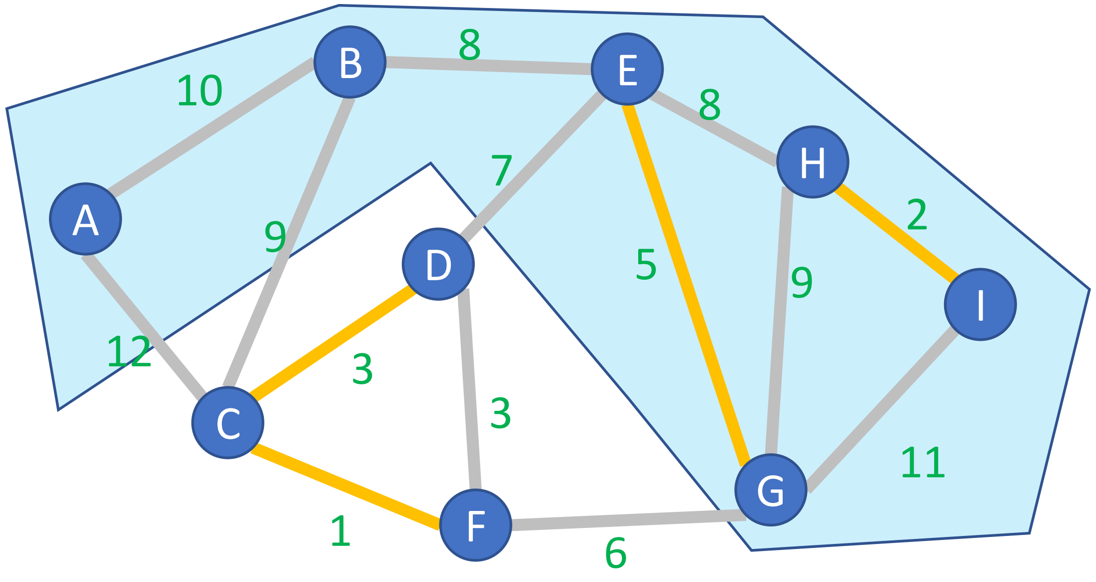

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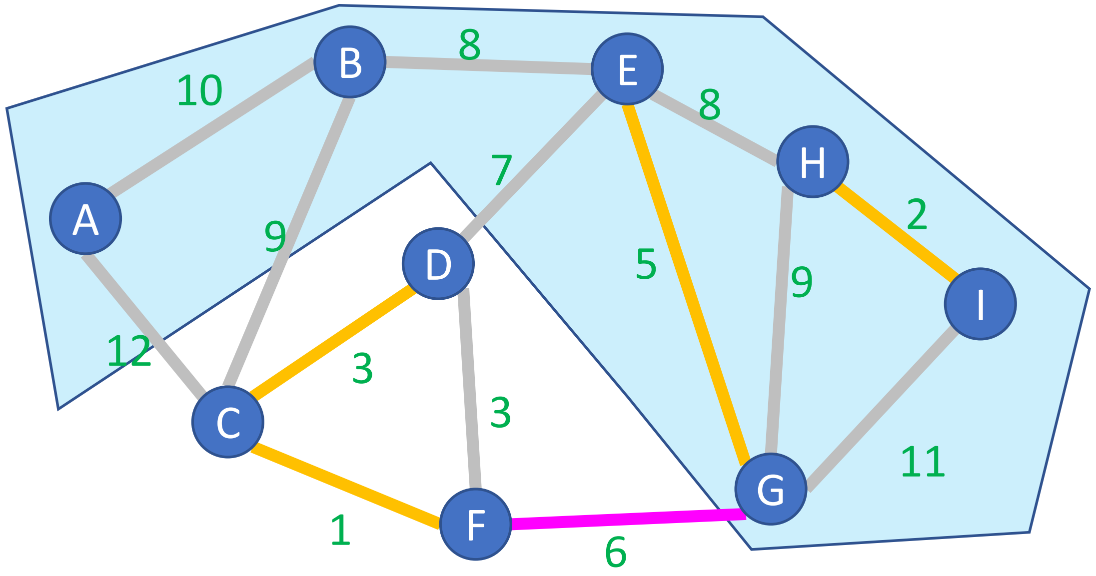

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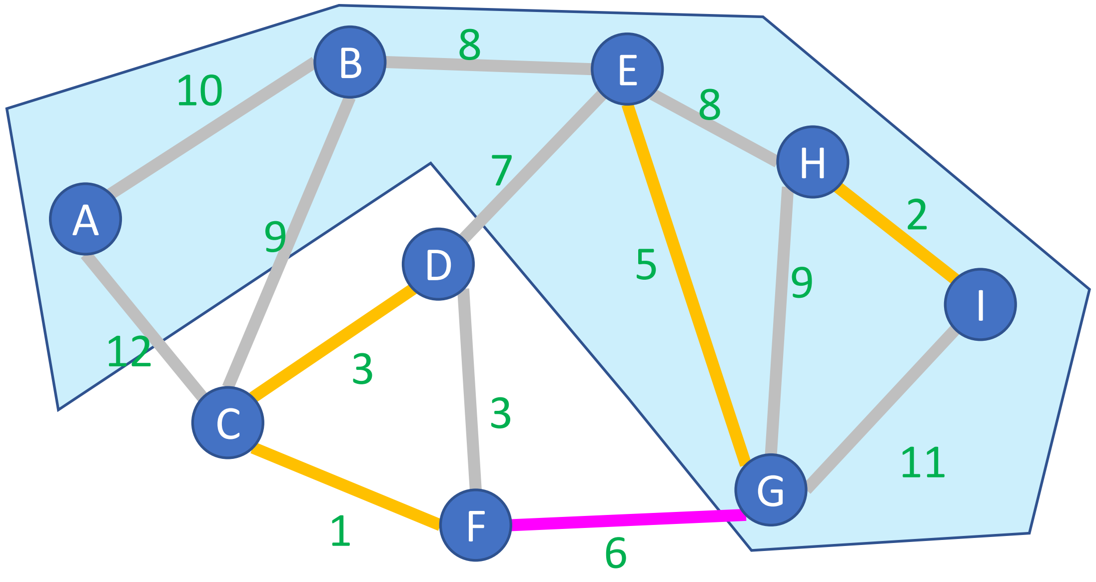
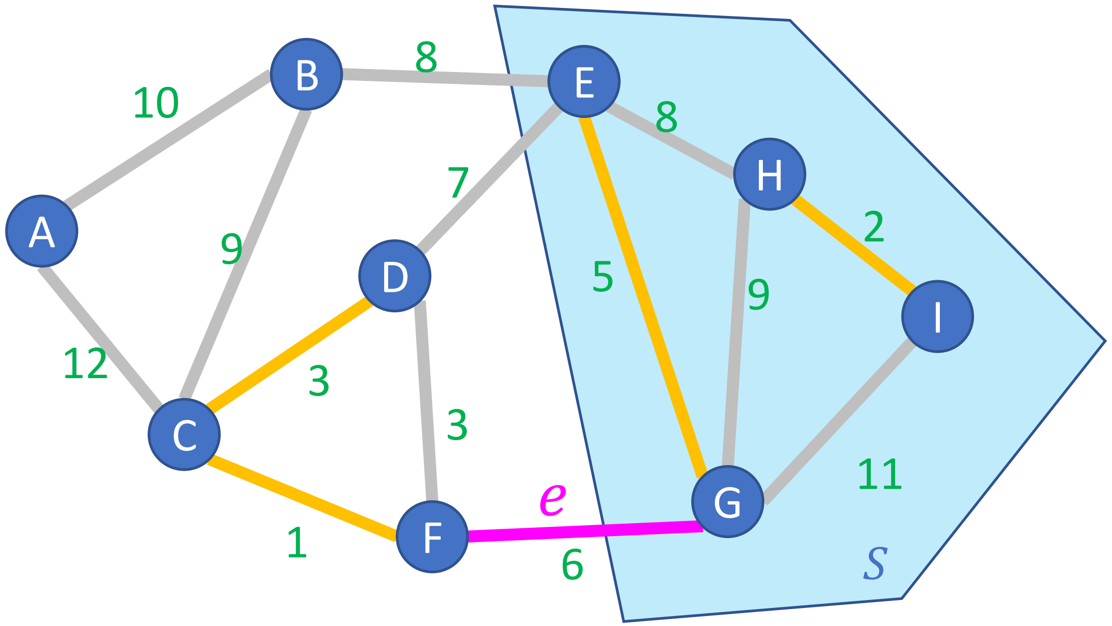


Proof of Kruskal’s Algorithm

Start with an empty tree A
Repeat $V - 1$ times:
Add the min-weight edge that doesn’t cause a cycle



Proof: Suppose we have some arbitrary set of edges A that Kruskal’s has already selected to include in the MST. $e = (F, G)$ is the edge Kruskal’s selects to add next

We know that there cannot exist a path from F to G using only edges in A because e does not cause a cycle

We can cut the graph therefore into 2 disjoint sets:

- nodes reachable from G using edges in A
- nodes reachable from F using edges in A

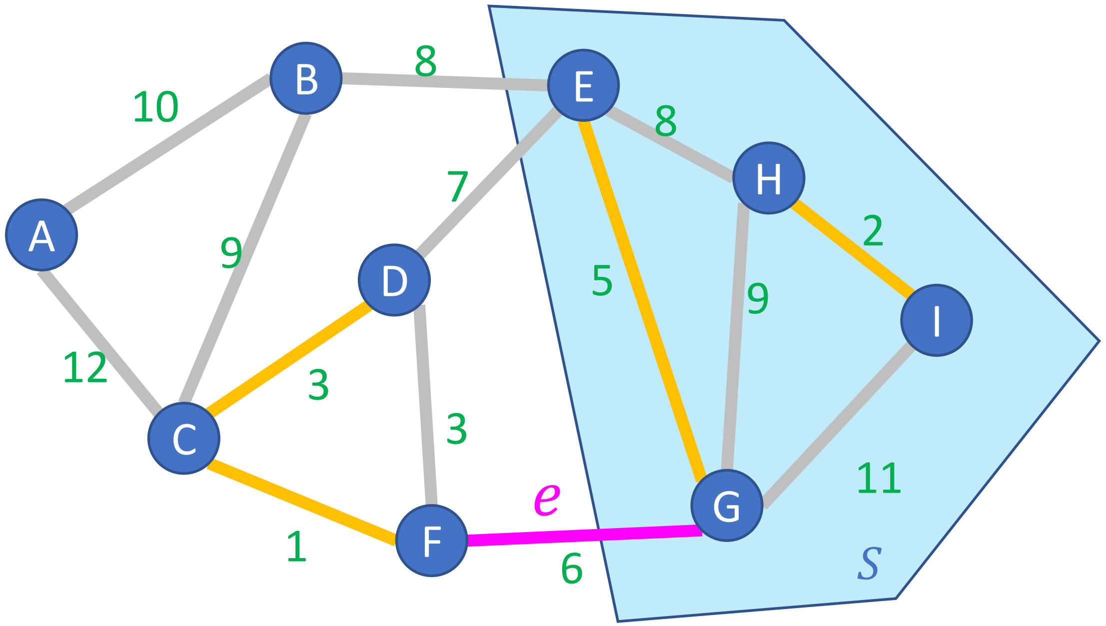
e is the minimum cost edge that crosses this cut, so by the Cut Theorem, Kruskal’s is optimal!

Kruskal’s Algorithm Runtime

Start with an empty tree A

Repeat $V - 1$ times:

Add the min-weight edge that doesn’t cause a cycle



Keep edges in a Disjoint-set data structure (very fancy)

$O(E \log V)$

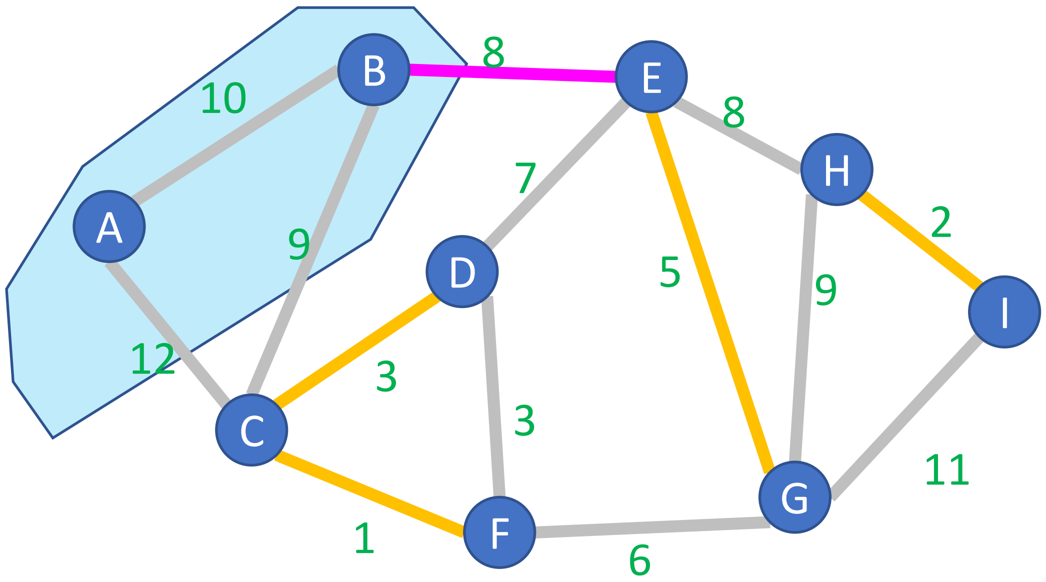
General MST Algorithm

Start with an empty tree A

Repeat $V - 1$ times:

Pick a cut $(S, V - S)$ which A respects

Add the min-weight edge which crosses $(S, V - S)$



Prim's Algorithm

Start with an empty tree A

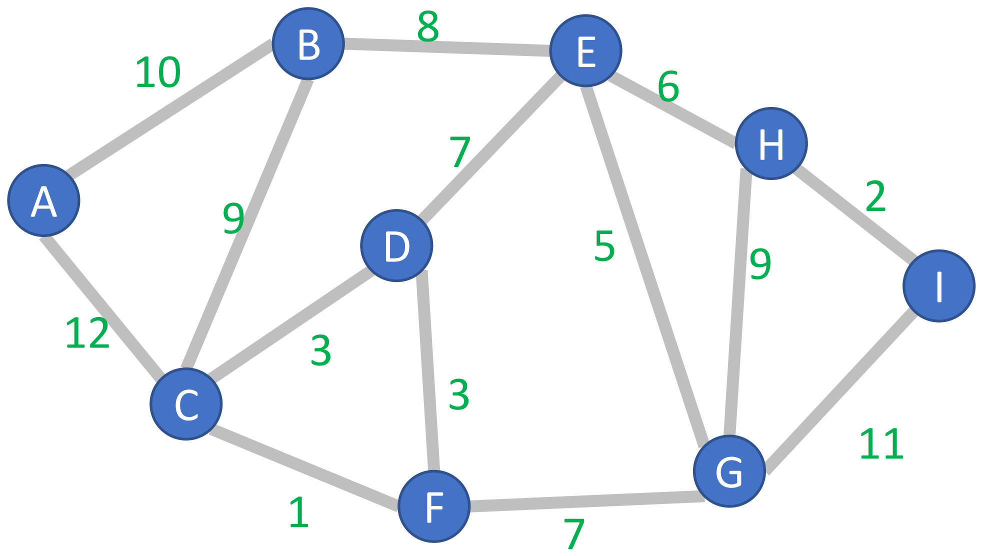
Repeat $V-1$ times:

Pick a cut $(S, V-S)$ which A respects

Add the min-weight edge which crosses $(S, V-S)$

S is all endpoint of edges in A

e is the min-weight edge that grows the tree



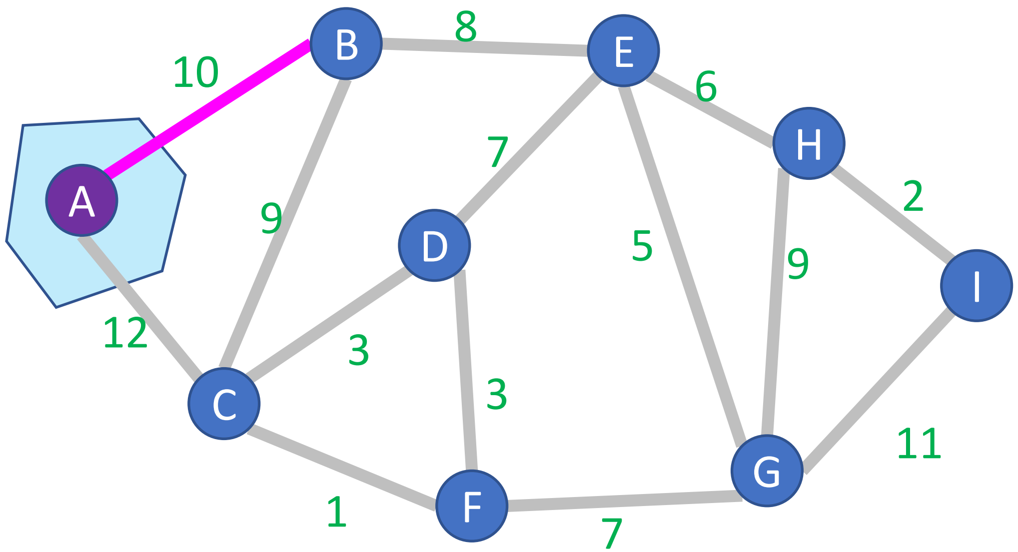
Prim's Algorithm

Start with an empty tree A

Pick a start node

Repeat $V - 1$ times:

Add the min-weight edge which connects to node in A with a node not in A



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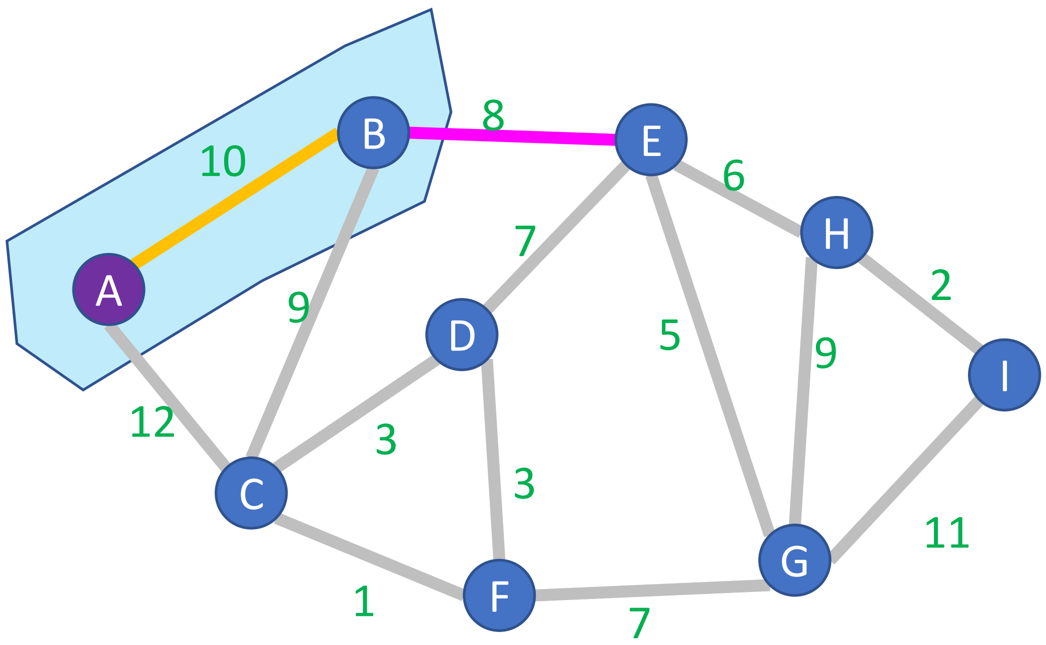

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Start with an empty tree A

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Repeat $V - 1$ times:

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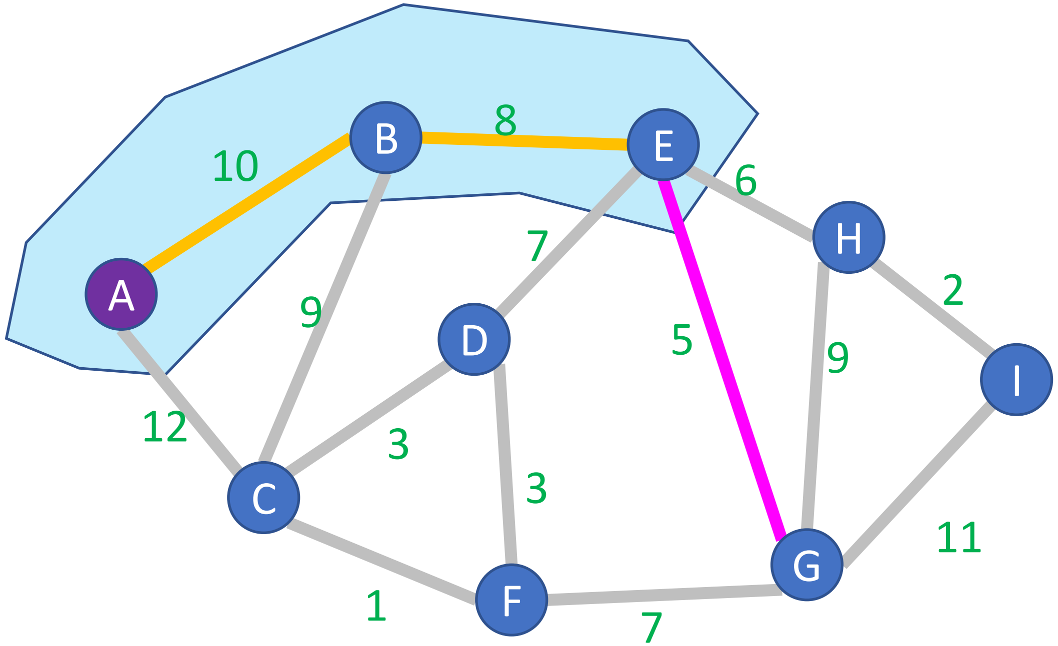

Prim's Algorithm

Start with an empty tree A

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Repeat $V - 1$ times:

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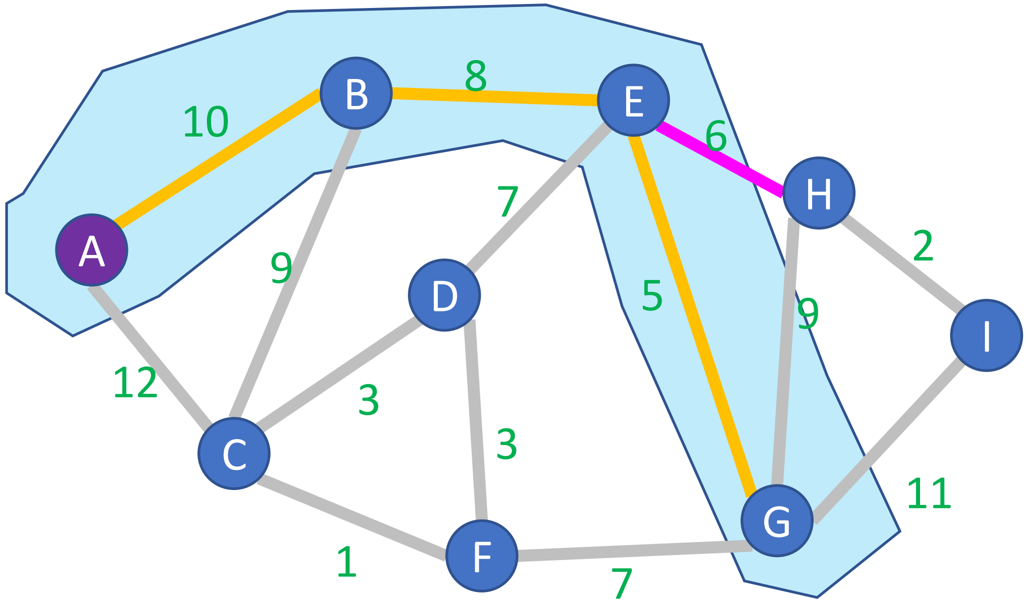


Prim's Algorithm

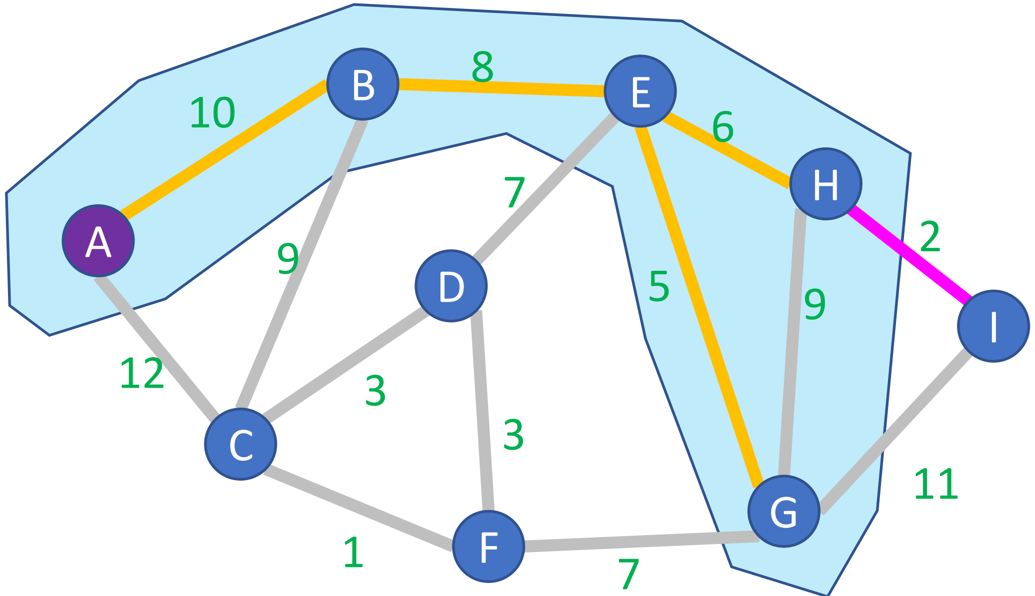
Start with an empty tree A

Pick a start node

Repeat $V - 1$ times:

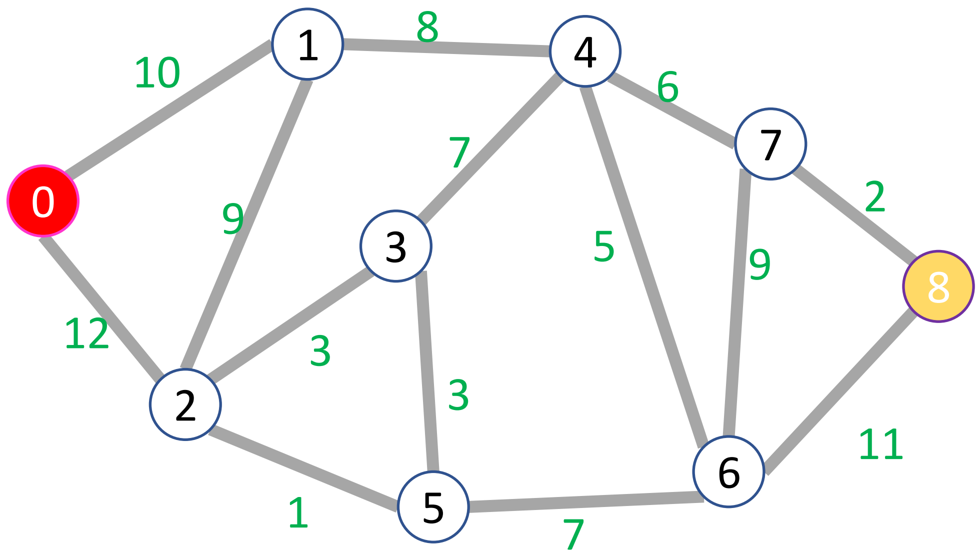
Add the min-weight edge which connects to node in A with a node not in A

Keep edges in a Heap
$O(E \log V)$



Dijkstra’s Algorithm

```
int dijkstras(graph, start, end){
        PQ = new minheap();
        PQ.insert(0, start);  // priority=0, value=start
        start.distance = 0;
        while (!PQ.isEmpty){
                current = PQ.extractmin();
                if (current.known){ continue;}
                current.known = true;
                for (neighbor : current.neighbors){
                        if (!neighbor.known){
                                new_dist = current.distance + weight(current,neighbor);
                                if(neighbor.dist != ∞){ PQ.insert(new_dist, neighbor);}
                                else if (new_dist < neighbor. distance){
                                        neighbor. distance = new_dist;
                                        PQ.decreaseKey(new_dist,neighbor); }
                        }
                }
        }
        return end.distance;
}
```



Prim’s Algorithm

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                if (current.known){ continue;}
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                for (neighbor : current.neighbors){
                        if (!neighbor.known){
                                new_dist = weight(current,neighbor);
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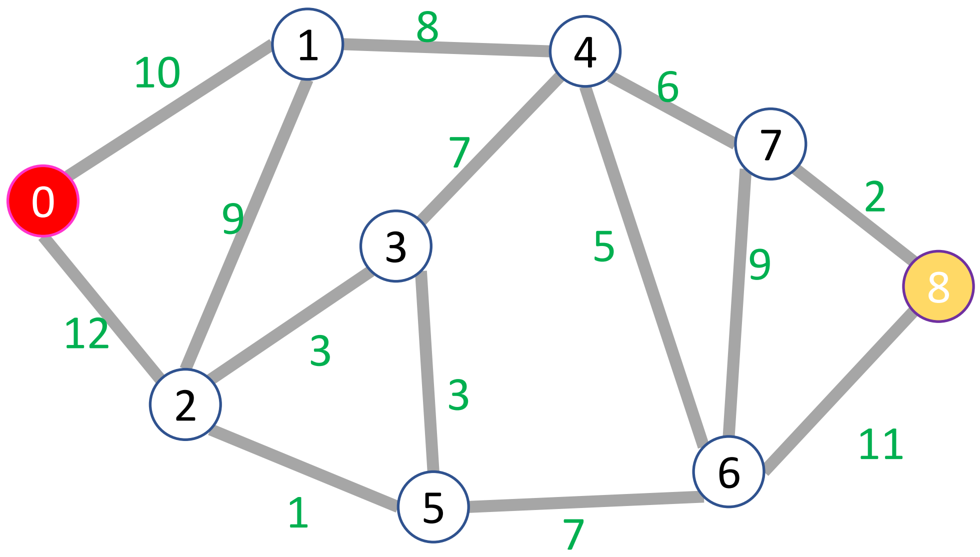
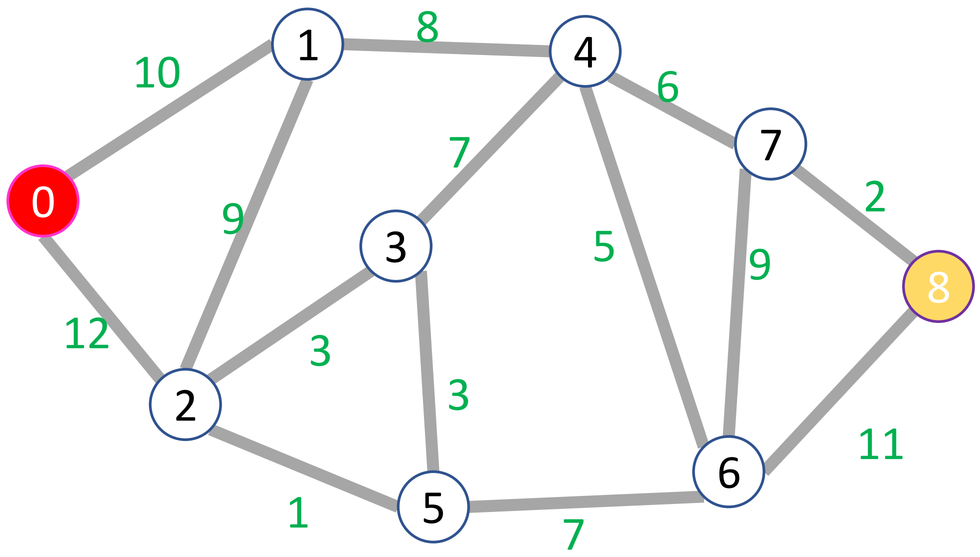
                                else if (new_dist < neighbor. distance){
                                        neighbor. distance = new_dist;
                                        PQ.decreaseKey(new_dist,neighbor); }
                        }
                }
        }
        return end.distance;
}
```



Dijkstra’s Algorithm

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                                else if (new_dist < neighbor. distance){
                                        neighbor. distance = new_dist;
                                        PQ.decreaseKey(new_dist,neighbor); }
                        }
                }
        }
        return end.distance;
}
```

Prim’s Algorithm

```
int dijkstras(graph, start, end){
	PQ = new minheap();
	PQ.insert(0, start);  // priority=0, value=start
	start.distance = 0;
	while (!PQ.isEmpty){
		current = PQ.extractmin();
		if (current.known){ continue;}
		current.known = true;
		for (neighbor : current.neighbors){
			if (!neighbor.known){
				new_dist = weight(current,neighbor);
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				else if (new_dist < neighbor. distance){
					neighbor. distance = new_dist;
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			}
		}
	}
	return end.distance;
}
```