CSE 332 Autumn 2023
Lecture 26: P & NP

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Tractability

n^r n^2 2^n

- Tractable:
  - Feasible to solve in the "real world"
- Intractable:
  - Infeasible to solve in the "real world"
- Whether a problem is considered "tractable" or "intractable" depends on the use case
  - For machine learning, big data, etc. tractable might mean $\mathrm{O}(n)$ or even $O(\log n)$
  - For most applications it's more like $O(n^3)$ or $O(n^2)$
- A strange pattern:
  - Most "natural" problems are either done in small-degree polynomial (e.g. n^2) or else exponential time (e.g. 2^n)
  - It's rare to have problems which require a running time of n^5, for example

Complexity Classes

- A Complexity Class is a set of problems (e.g. sorting, Euler path, Hamiltonian path)
  - The problems included in a complexity class are those whose most efficient algorithm has a specific upper bound on its running time (or memory use, or…)
- Examples:
  - The set of all problems that can be solved by an algorithm with running time $O(n)$
    - Contains: Finding the minimum of a list, finding the maximum of a list, buildheap, summing a list, etc.
  - The set of all problems that can be solved by an algorithm with running time $O(n^2)$
    - Contains: everything above as well as sorting, Euler path
  - The set of all problems that can be solved by an algorithm with running time $O(n!)$
    - Contains: everything we've seen in this class so far

Complexity Classes and Tractability

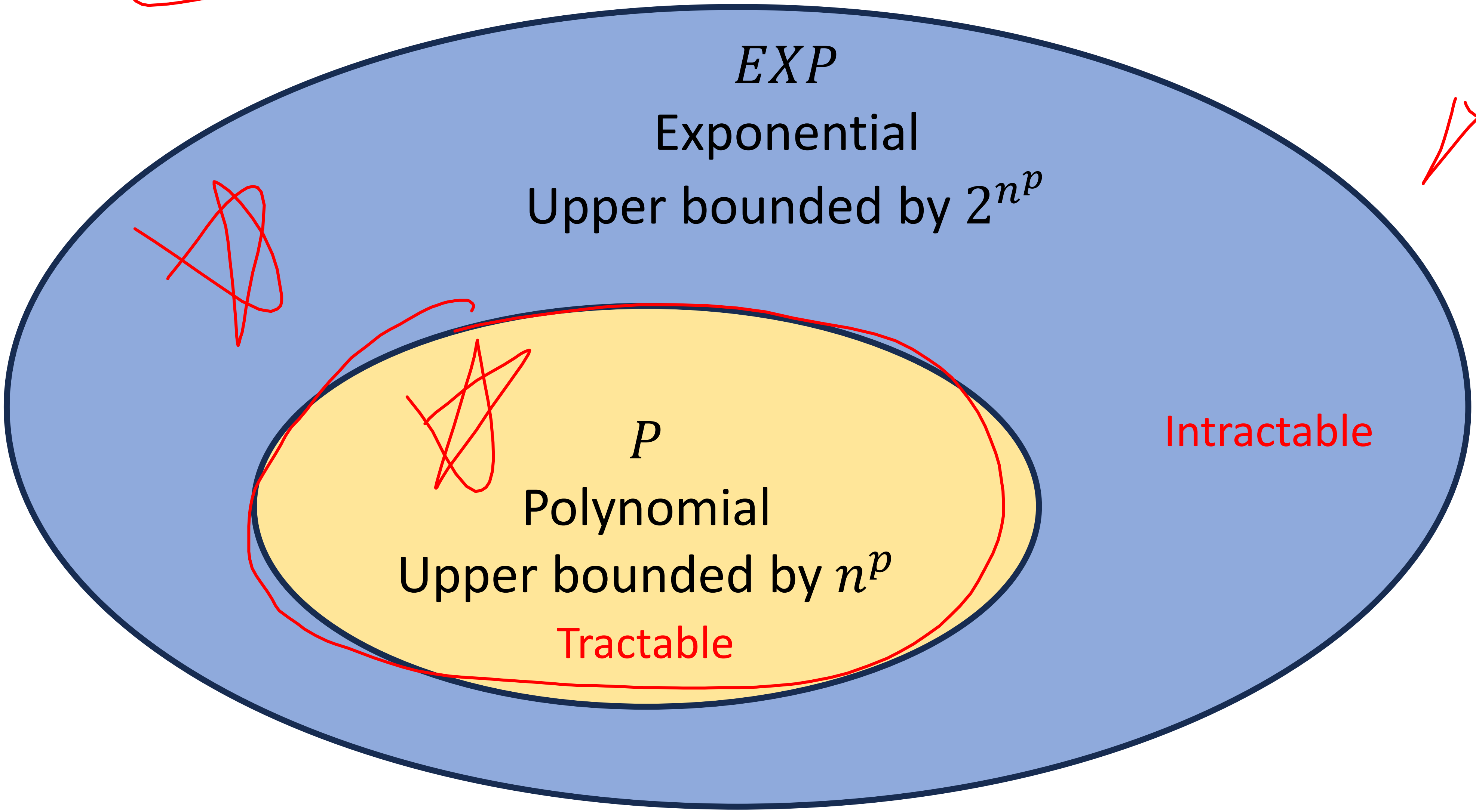
- To explore what problems are and are not tractable, we give some complexity classes special names:
- Complexity Class P:
  - Stands for "Polynomial"
  - The set of problems which have an algorithm whose running time is $O(n^p)$ for some choice of $p \in \mathbb{R}$.
  - We say all problems belonging to P are "Tractable"
- Complexity Class EXP:
  - Stands for "Exponential"
  - The set of problems which have an algorithm whose running time is $O(2^{n^p})$ for some choice of $p \in \mathbb{R}$
  - We say all problems belonging to $EXP - P$ are "Intractable"
    - Disclaimer: Really it's all problems outside of P, and there are problems which do not belong to EXP, but we're not going to worry about those in this class

n^2 n^3 n^{17}

EXP and P

Important!
$P \subset EXP$
Every problem within P is also within EXP
The intractable ones are the problems within EXP but NOT P

$P \subset Exp$

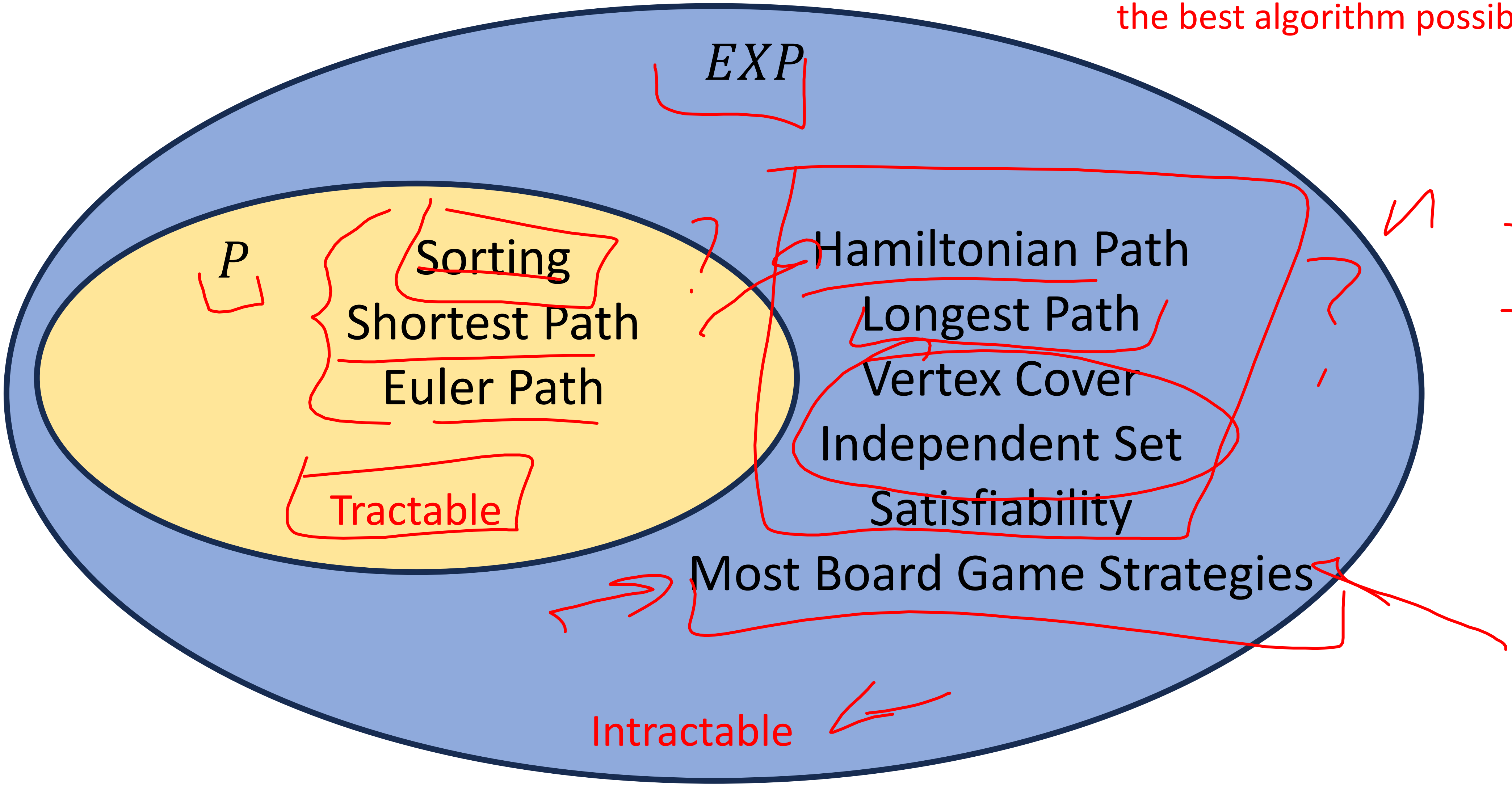


Members

Important!

Some of the problems listed in EXP could also be members of P

Since membership is determined by a problem's *most* efficient algorithm, knowing if a problem belongs to P requires knowing the best algorithm possible!



Studying Complexity and Tractability

- Organizing problems into complexity classes helps us to reason more carefully and flexibly about tractability
- The goal for each problem is to either
  - Find an efficient algorithm if it exists
    - i.e. show it belongs to P
  - Prove that no efficient algorithm exists
    - i.e. show it does not belong to P
- Complexity classes allow us to reason about sets of problems at a time, rather than each problem individually
  - If we can find more precise classes to organize problems into, we might be able to draw conclusions about the entire class
  - It may be easier to show a problem belongs to class C than to P, so it may help to show that $C \subseteq P$

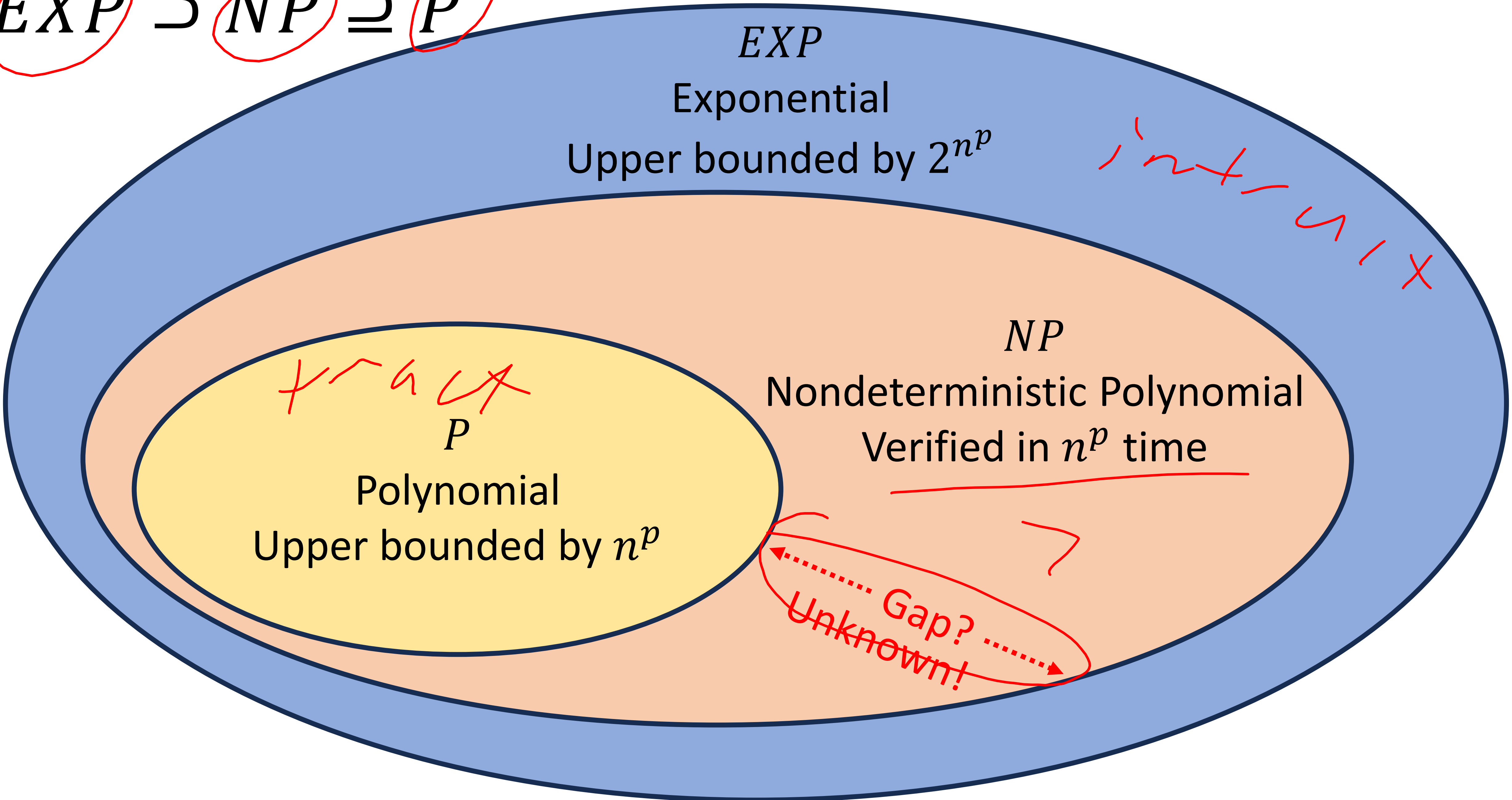
Some problems in EXP seem “easier”

- There are some problems that we do not have polynomial time algorithms to solve, but provided answers are easy to check
- Hamiltonian Path:
  - It’s “hard” to look at a graph and determine whether it has a Hamiltonian Path
  - It’s “easy” to look at a graph and a candidate path together and determine whether THAT path is a Hamiltonian Path
    - It’s easy to **verify** whether a given path is a Hamiltonian path

Class NP

- NP
  - The set of problems for which a candidate solution can be verified in polynomial time
  - Stands for “Non-deterministic Polynomial”
    - Corresponds to algorithms that can guess a solution (if it exists), that solution is then verified to be correct in polynomial time
    - Can also think of as allowing a special operation that allows the algorithm to magically guess the right choice at each step of an exhaustive search
- $P \subseteq NP$
  - Why?

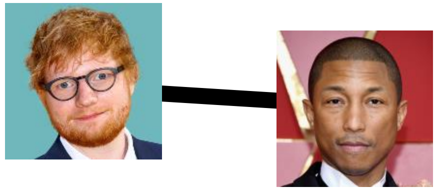
$EXP \supset NP \supseteq P$



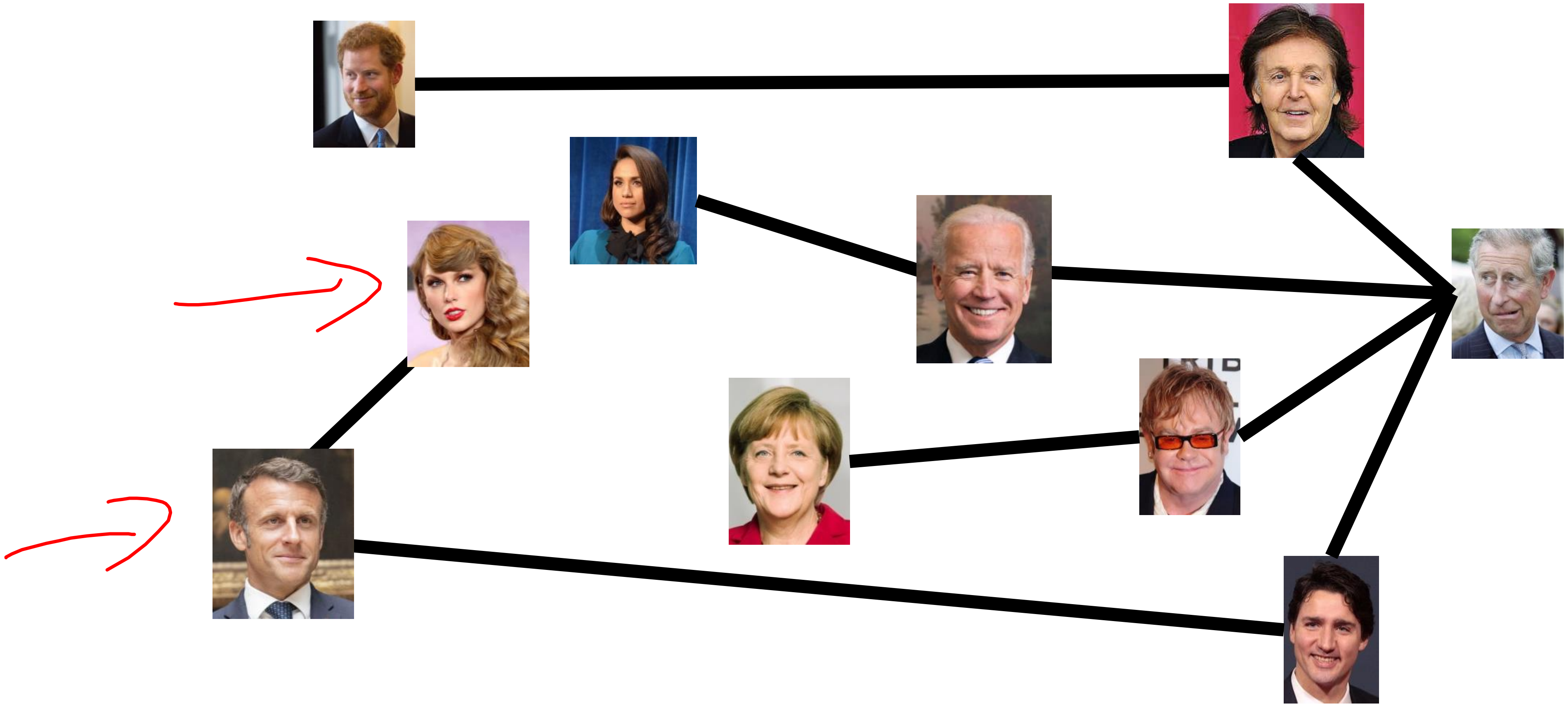
Solving and Verifying Hamiltonian Path $\in NP$

- Give an algorithm to solve Hamiltonian Path
  - Input: $G = (V, E)$
  - Output: True if G has a Hamiltonian Path
  - Algorithm: Check whether each permutation of V is a path.
    - Running time: $|V|!$, so does not show whether it belongs to P
- Give an algorithm to verify Hamiltonian Path
  - Input: $G = (V, E)$ and a sequence of nodes
  - Output: True if that sequence of nodes is a Hamiltonian Path
  - Algorithm:
    - Check that each node appears in the sequence exactly once
    - Check that the sequence is a path
    - Running time: $O(V \cdot E)$, so it belongs to NP

Party Problem



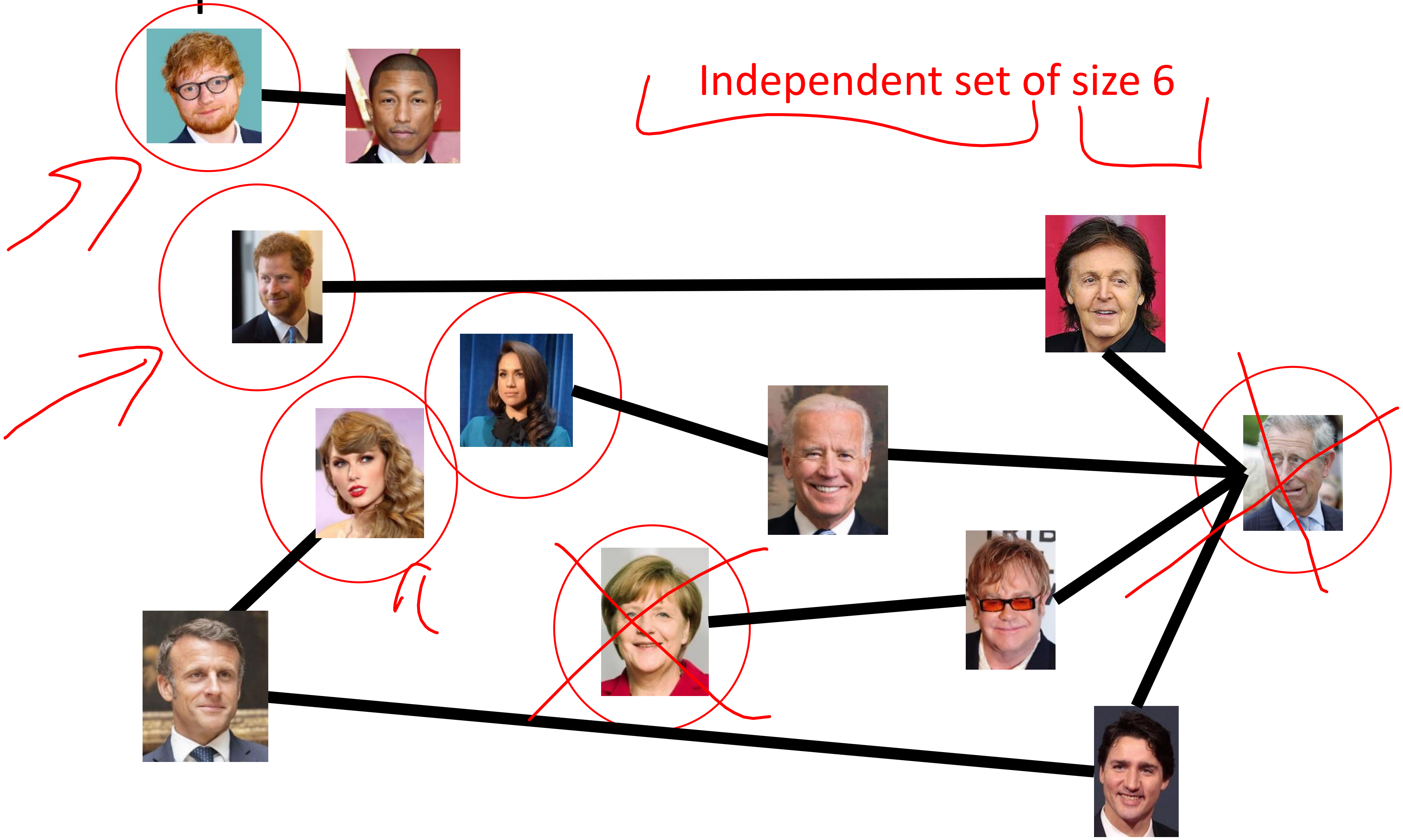
Draw Edges between people who don't get along
How many people can I invite to a party if everyone must get along?



Independent Set

- Independent set:
  - $S \subseteq V$ is an independent set if no two nodes in S share an edge
- Independent Set Problem:
  - Given a graph $G = (V, E)$ and a number k, determine whether there is an independent set S of size k

Example



Solving and Verifying Independent Set

$2^{|V|}$

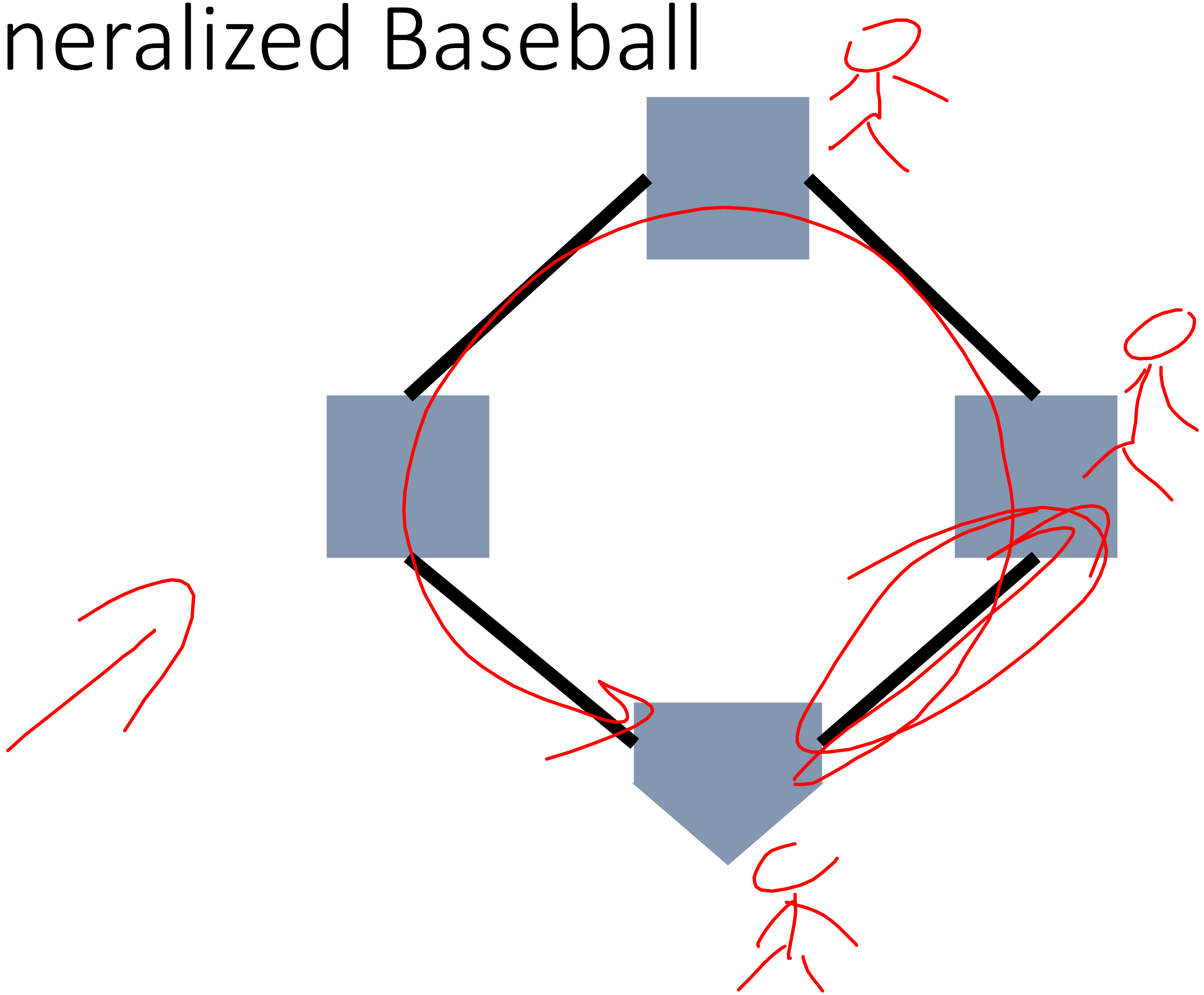
- Give an algorithm to solve independent set
    - Input: $G = (V, E)$ and a number k
    - Output: True if G has an independent set of size k

$P(V)$

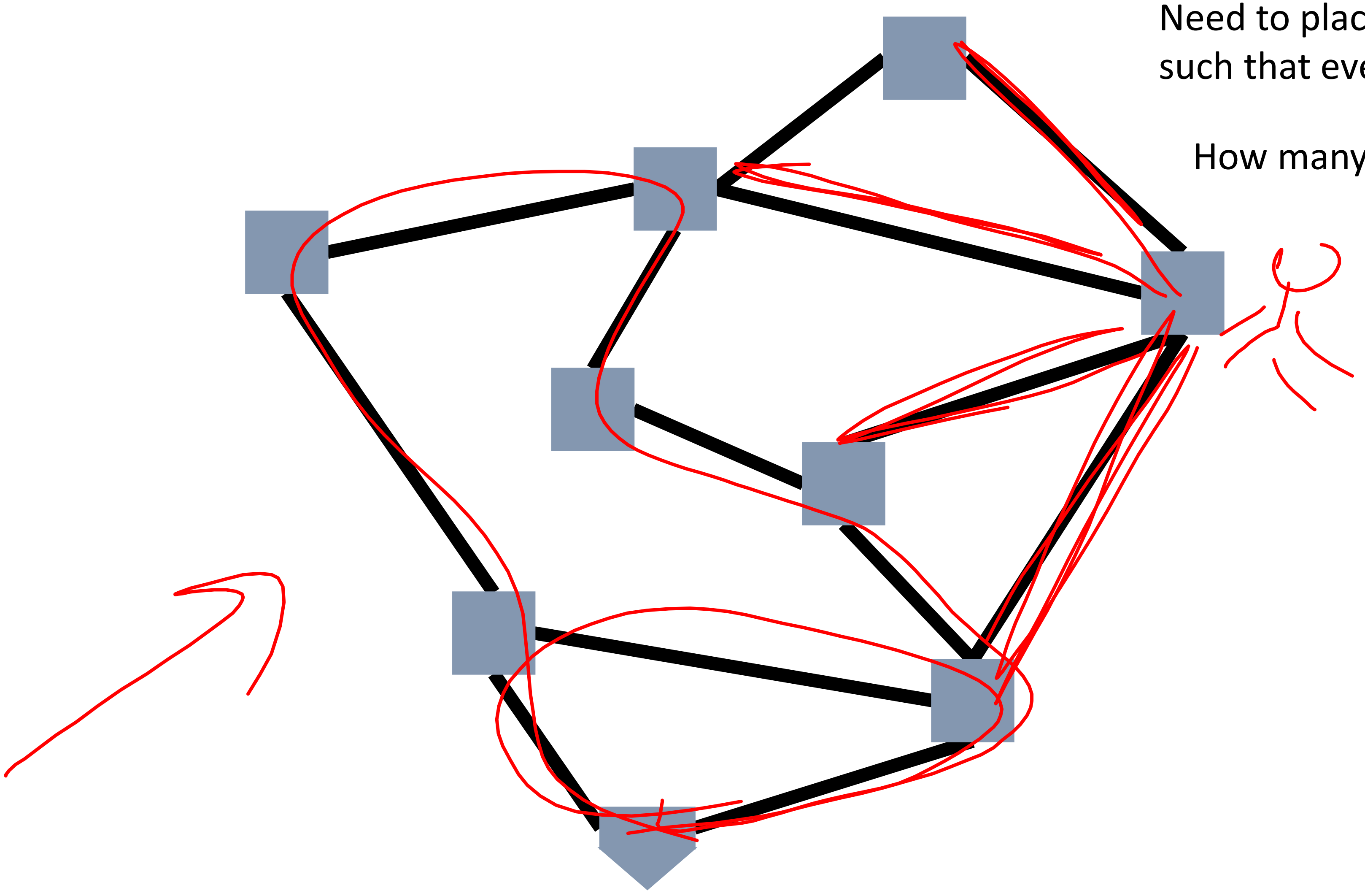
- Give an algorithm to verify independent set
    - Input: $G = (V, E)$, a number k, and a set $S \subseteq V$
    - Output: True if S is an independent set of size k

$n^2 \cdot E$

Generalized Baseball



Generalized Baseball



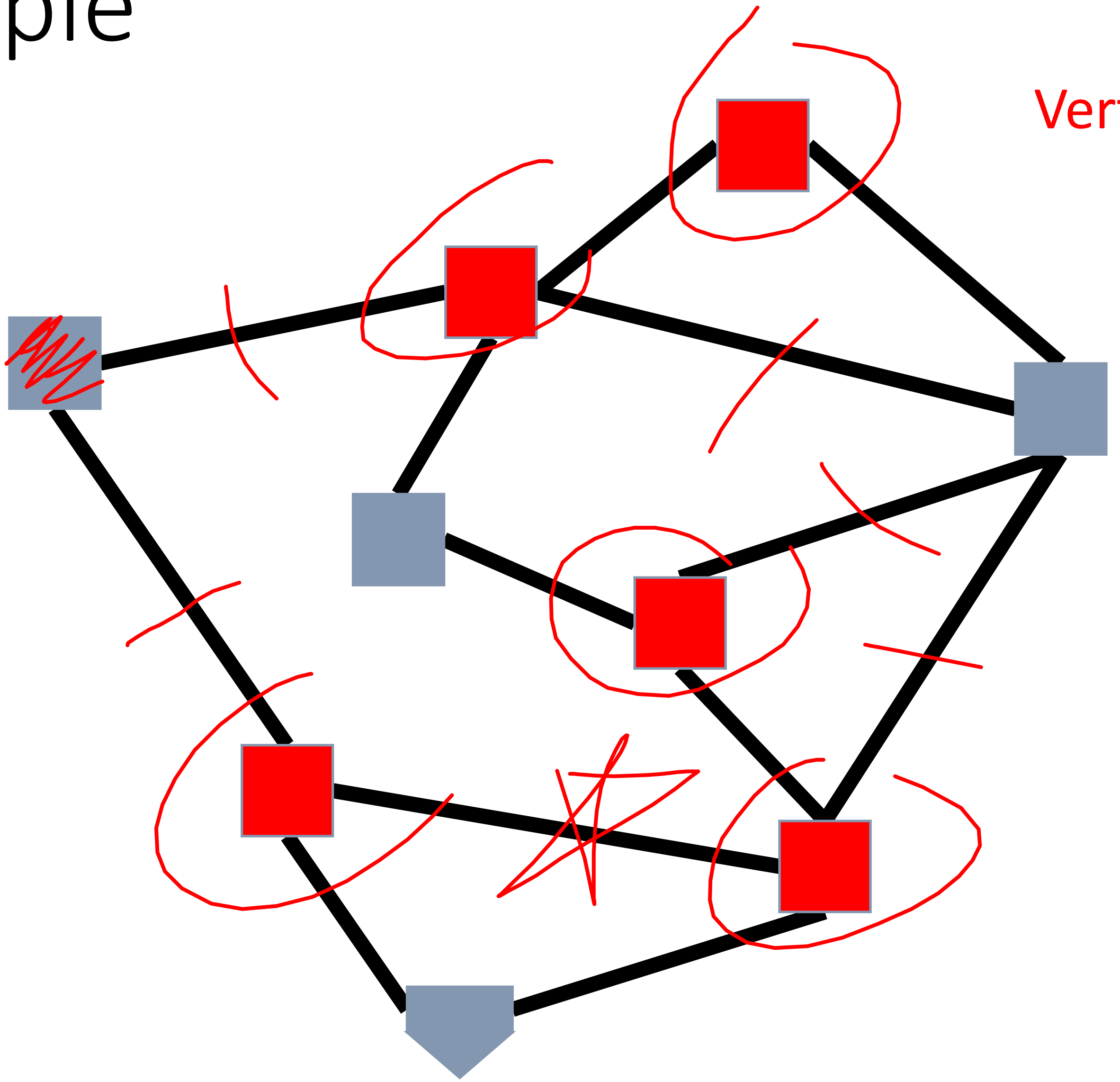
Need to place defenders on bases such that every edge is defended

How many defenders would suffice?

Vertex Cover

- Vertex Cover:
  - $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- Vertex Cover Problem:
  - Given a graph $G = (V, E)$ and a number k, determine if there is a vertex cover C of size k

Example



Vertex cover of size 5

Solving and Verifying Vertex Cover $\in NP$

- Give an algorithm to solve vertex cover
  - Input: $G = (V, E)$ and a number k
  - Output: True if G has a vertex cover of size k

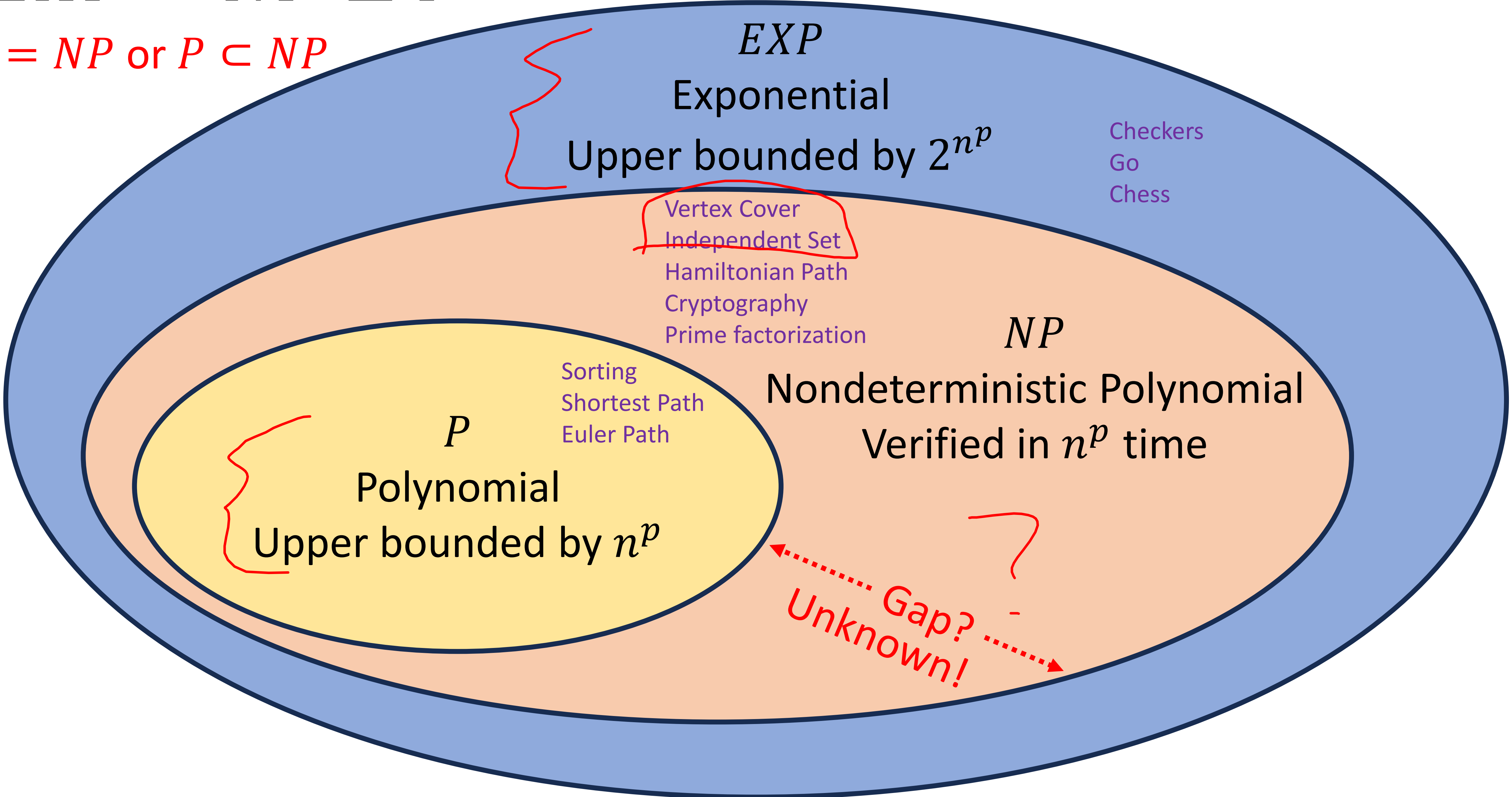
$2^{(V)}$

- Give an algorithm to verify vertex cover
  - Input: $G = (V, E)$,a number k, and a set $S \subseteq E$
  - Output: True if S is a vertex cover of size k

$E \cdot k$

$EXP \supset NP \supseteq P$

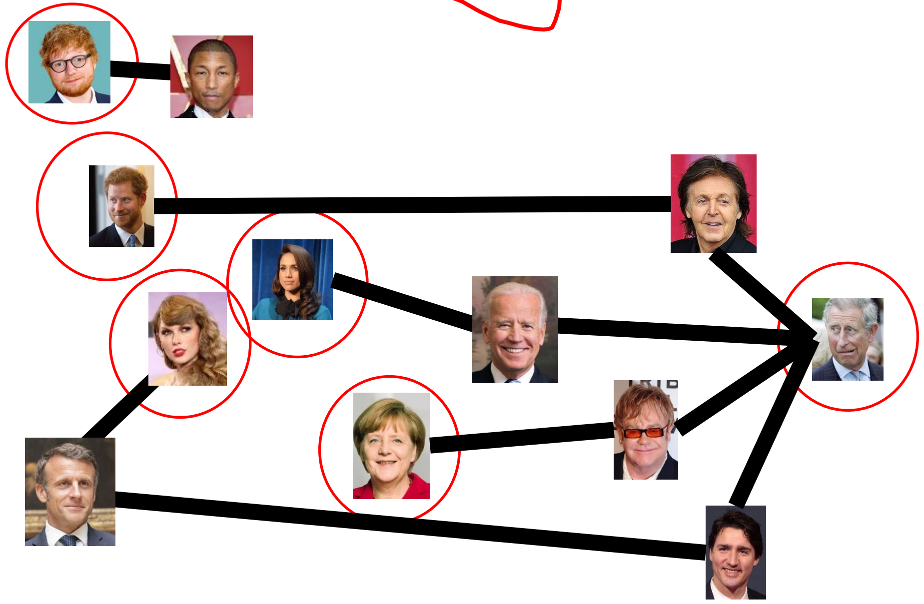
$P = NP$ or $P \subset NP$



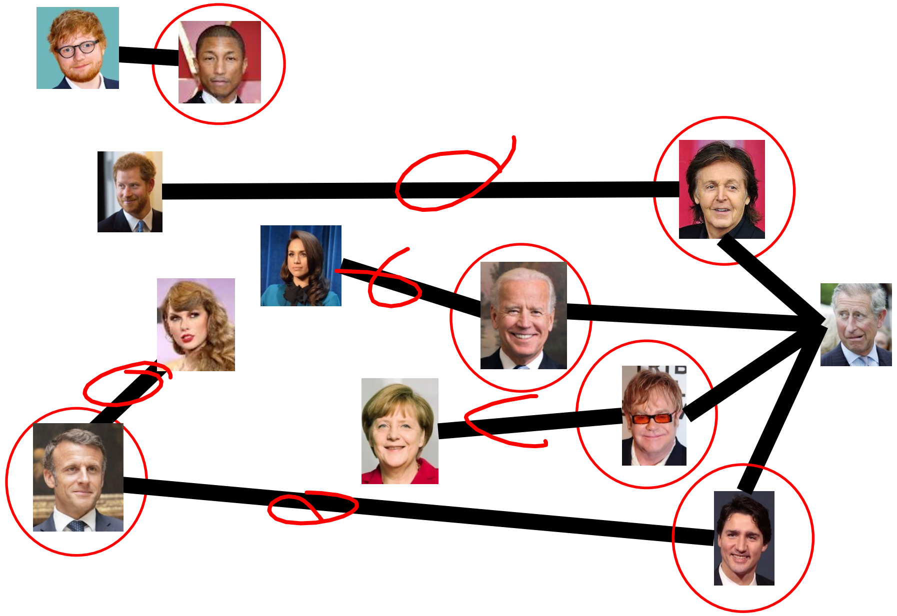
Way Cool!

S is an independent set of G iff $V - S$ is a vertex cover of G

Independent Set

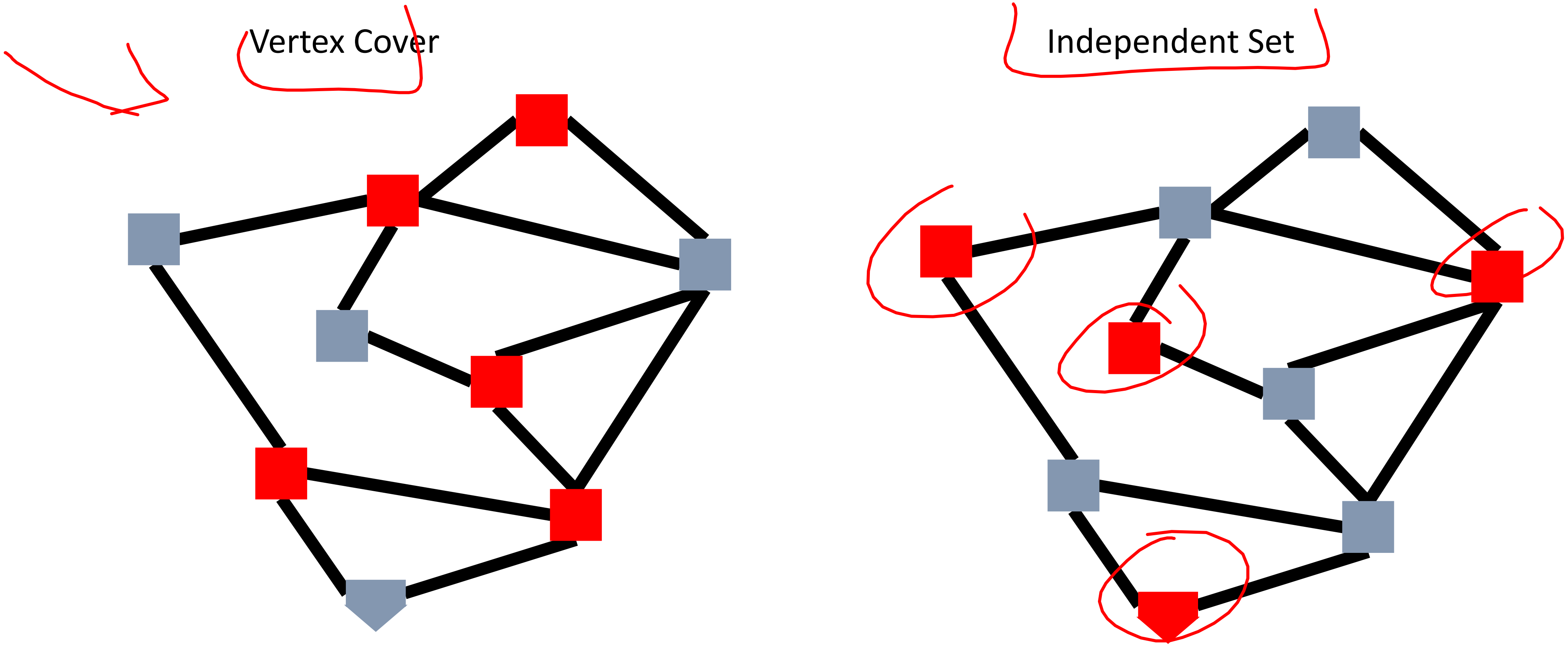


Vertex Cover



Way Cool!

S is an independent set of G iff $V - S$ is a vertex cover of G



Solving Vertex Cover and Independent Set

- Algorithm to solve vertex cover
  - Input: $G = (V, E)$ and a number k
  - Output: True if G has a vertex cover of size k
    - Check if there is an Independent Set of G of size $|V| - k$
- Algorithm to solve independent set
  - Input: $G = (V, E)$ and a number k
  - Output: True if G has an independent set of size k
    - Check if there is a Vertex Cover of G of size $|V| - k$

Either both problems belong to P, or else neither does!

NP-Complete

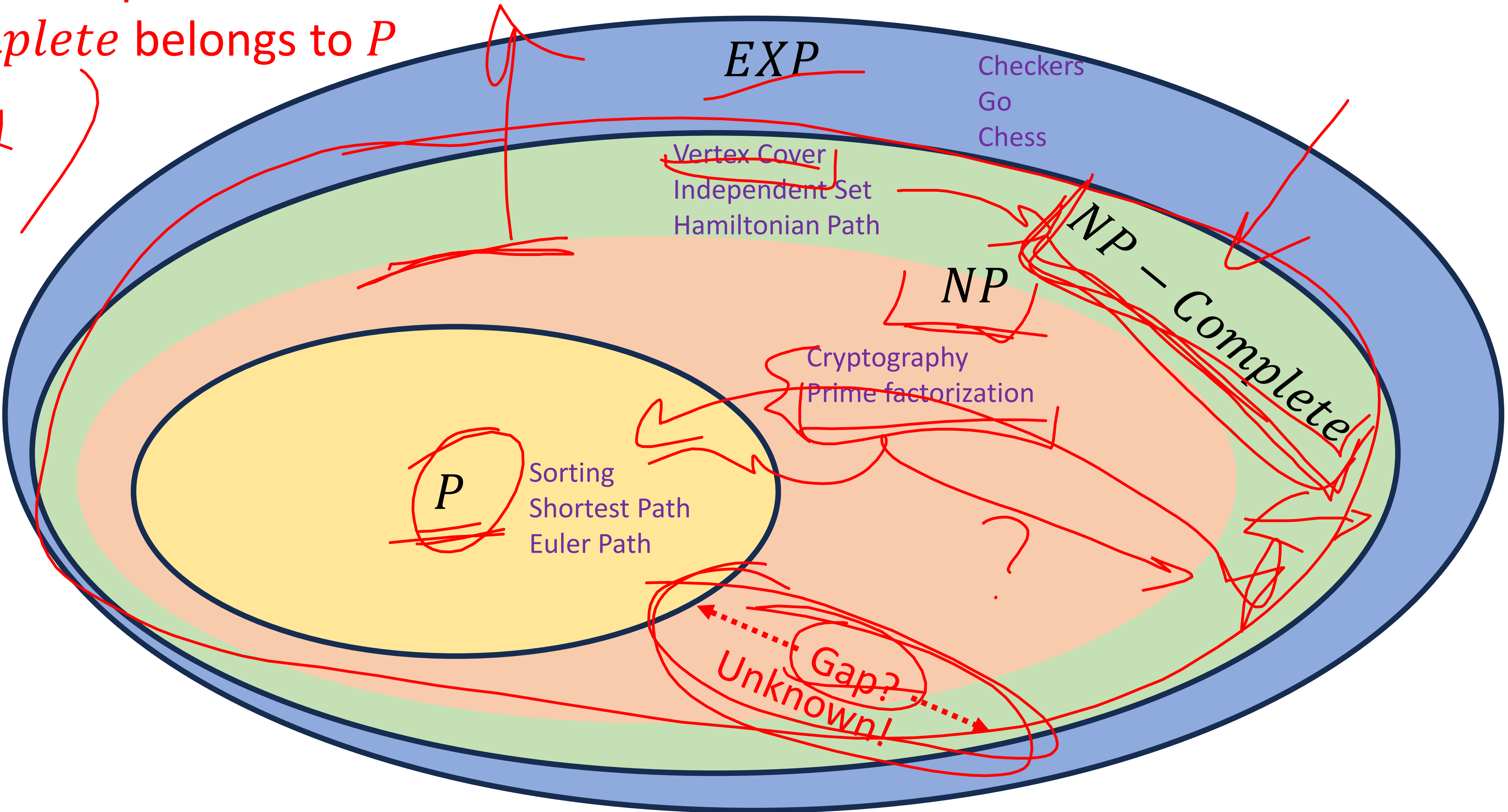
- A set of "together they stand, together they fall" problems
- The problems in this set either all belong to P, or none of them do
- Intuitively, the "hardest" problems in NP
- Collection of problems from NP that can all be "transformed" into each other in polynomial time
  - Like we could transform independent set to vertex cover, and vice-versa
  - We can also transform vertex cover into Hamiltonian path, and Hamiltonian path into independent set, and …

$EXP \supset NP - Complete \supseteq NP \supseteq P$

$P = NP$ iff some problem from $NP - Complete$ belongs to P

(NP-H)

¬NP



Overview

- Problems not belonging to P are considered intractable
- The problems within NP have some properties that make them seem like they might be tractable, but we've been unsuccessful with finding polynomial time algorithms for many
- The class $NP - Complete$ contains problems with the properties:
  - All members are also members of NP
  - All members of NP can be transformed into every member of $NP - Complete$
  - Therefore if any one member of $NP - Complete$ belongs to P, then $P = NP$

Why should YOU care?

- If you can find a polynomial time algorithm for any $NP - Complete$ problem then:
  - You will win $1million
  - You will win a Turing Award
  - You will be world famous
  - You will have done something that no one else on Earth has been able to do in spite of the above!
- If you are told to write an algorithm a problem that is $NP - Complete$
  - You can tell that person everything above to set expectations
  - Change the requirements!
  - **Approximate the solution**: Instead of finding a path that visits every node, find a path that visits at least 75% of the nodes
  - **Add Assumptions**: problem might be tractable if we can assume the graph is acyclic, a tree
  - **Use Heuristics**: Write an algorithm that's "good enough" for small inputs, ignore edge cases