

In this assignment we'll look at Boolean expressions. This will be great practice for impressing me when I go dressed as Mr. Spock to the next Star Trek convention. er, I mean, when my *friend* goes dressed as Mr. Spock. . . um, anyway, why don't I just explain the assignment.

Part I

Boolean expressions are what are used in, for example, `if` statement predicates. The following are Boolean expressions:

```
true And (false Or Not false)
Not x And (true Or x)
```

Because the first expression contains only the constants `true` and `false`, we say it is an expression over constants. It *evaluates* to `true`. The second expression contains a variable `x`; we have to *bind* a value to `x` before we can evaluate the expression. In a little abuse of terminology, we say that `x` is *free* in the expression. We can *bind* free variables to constants, for example binding `x` to `true` in the expression `(Not x And (true Or x))` gives `Not true And (true Or true)`, which evaluates to `false`. If we bound `x` to `false`, the expression would evaluate to `true`.

We say that an expression is *constant* if it contains no free variables. A constant expression can be evaluated without having to bind anything.

To express a Boolean expressions in ML, we will use the following data type.

```
datatype 'a expr =
  Const of bool
  | Var of 'a
  | Not of 'a expr
  | And of 'a expr * 'a expr
  | Or of 'a expr * 'a expr
exception UnboundVar
```

For the first part of this assignment, we'll use only `string expr`. Thus, the ML expression that's the same as the second example above is `And(Not(Var "x"),Or(Const true, Var "x"))`.

1. Write a function `free_vars` that takes an expression and return a `string list` of any a free variables. Variables should not be repeated if they appear multiple times; use the set functions from the last homework (and include them in your turnin).
2. Write a function `eval` that takes a constant expression and returns its value. If `eval` is passed a function with free variables, it should raise an `UnboundVar` exception.

Note: a natural way to solve this problem will evaluate `Or(Const true,Var "x")` as `true` even though it's not really a constant expression. That's okay. The same solution will raise an exception if passed `Or(Var "x",Const true)`, even though it's the same expression as before. That's okay too. The point of this problem is to get practice in evaluating things, not figuring out the type of an expression.

3. Write a function `bind1` that takes a variable, a Boolean value and an expression, and returns an expression with all instances of the variable bound to the truth value. It should be curried function. For example,

```
bind1 "x" true (And(Var "x",Or(Var "y", Var "x")))
--> And(Const true,Or(Var "y", Const true))
bind1 "x" true (And(Var "z",Or(Var "y", Var "z")))
--> And(Var "z",Or(Var "y", Var "z"))
```

4. `bind1` is nice, but not very general. For example, we couldn't use it to change some variable in an expression to a different variable. Write a function `bind` that takes a function `binder` and an expression, applies `binder` to the argument of a `Var`, and leaves the rest of the expression unchanged. `binder` should take the value from a `Var` and return an expression. Then write functions `bindvar` and `changevar` using `bind` so that `bindvar` is equivalent to `bind1`, and `changevar` is similar except that variables are changed. For example,

```
bindvar "x" true (And(Var "x",Or(Var "y", Var "x")))
--> And(Const true,Or(Var "y", Const true))
changevar "x" "help" (And(Var "x",Or(Var "y", Var "x")))
--> (And(Var "help",Or(Var "y", Var "help")))
```

5. Write a function `satisfying_assignments` that takes an expression and returns a `(string*bool)` list `list` of all satisfying assignments, if any exist. For example,

```
satisfying_assignments (And(Not(Var "x"),Or(Const true, Var "x")))
--> [{"x",false}]
satisfying_assignments (And(Not(Var "x"),Or(Const false, Var "x")))
--> nil
satisfying_assignments (Or(Var "x", Var "y"))
--> [{"x",true}, {"y",true}], [{"x",true}, {"y",false}],
    [{"x",false}, {"y",true}]]
satisfying_assignments (Const true)
--> [[]]
satisfying_assignments (Const false)
--> []
```

Hint: write a local function that takes an expression and a list of its free variables. If the list is empty, evaluate the expression; otherwise recurse, using `map` and `append` to build the new bindings lists. My implementation of this function is 11 lines.

Part II

It might be hard to believe now, but at some point just evaluating Boolean expressions will seem like a sort of dull thing to do. An exciting application of Boolean expressions is knowledge representation and planning in artificial intelligence.

As I write this, I'm at home. I'm hungry. Omlettes are crossing my mind. Omlettes require eggs. I don't have eggs. To buy eggs, I need to be where eggs are sold. QFC¹ sells eggs... How can I

¹the University of Washington makes no endorsement of companies, goods, services or products mentioned on this page.

represent this complex and seemingly amorphous collection of wisdom in an orderly way so that a computer could do something with it? Boolean logic to the rescue! The Boolean expressions above contain only Boolean variables and constants. To reason about the world, we introduce *predicates*, which are Boolean-valued functions of (typically) non-Boolean variables. For example, suppose I have a predicate `At` over a *domain* of locations including `Home`, `QFC` and `Ace`². Then in my current state `At(Home)` is true, but `At(QFC)` and `At(Ace)` are false. Similarly, I might represent knowledge about availability of goods using another predicate `Sells`, whose domain is (locations * products). For example, `Sells(QFC,Eggs)` and `Sells(Ace,Hammers)`.

In general, the world is described by a *state*, which is a list of predicates. All predicates in the state are true, and any others are false. For example, a state may be `[At(home), Sells(QFC, Eggs), Sells(Ace, Hammers)]`.

Returning to that which is uppermost in my mind, can I buy some eggs to make my omlette? Unfortunately not, because I'm at home and the eggs are at QFC, and a general principle in the real world³ is that I can't buy something unless I'm at a place that sells it. Again, I can codify this principle by saying the *action* `BUY(place, item)` has a *precondition* that must be fulfilled before it can happen, namely `At(place)` and `Sells(place, item)`.

I'm getting hungrier all the time, but we're closer to having some homework problems: here we have a Boolean expression, but the basic terms in it are predicates, rather than simple Boolean variables like `x` and `y` that we had before. We need one more concept. There's an important difference between the predicates `At(QFC)` and `At(place)`. The former, involving the location constant `QFC` is called a *ground predicate*, and in our application we'll always know whether each ground predicate is true or false. (In particular, the *state* contains only ground predicates.) On the other hand, we can't evaluate "`At(place)`", involving the location *variable* `place` without knowing a *binding* of `place` to an actual location. We call `QFC` a *constant atom* and `place` a *variable atom*. We'll represent the difference more explicitly when we implement this in ML, but in this write-up, we'll use the convention that constant atoms start with an upper-case letter and variable atoms start with a lower-case letter: `QFC`, `Eggs` and `Ace` are constant atoms; `place` and `item` are variable atoms.

So, can I buy eggs? More generally, what can I buy right now? This is equivalent to asking to find all possible bindings of `x` to locations and `y` to items such that the precondition of `BUY(x,y)` is satisfied by my current state. So that's our goal—given a *predicate* `expr` (as opposed to *string* `expr` as before) representing a precondition like the one for `BUY(x,y)`, and a state, find all bindings of variables that satisfy the expression.

For example, take the following state.

```
start = [ At(Home), Sells(QFC, Eggs), Sells(QFC, Bread), Sells(Ace, Hammers) ]
```

Then the set of all assignments to `x` and `y` satisfying the precondition `At(x)` and `Sells(x,y)` is empty, but if I am in the following state

```
start = [ At(QFC), Sells(QFC, Eggs), Sells(QFC, Bread), Sells(Ace, Hammers) ]
```

then the set of satisfying bindings for the precondition is

```
[[ (x, QFC), (y, Eggs) ], [ (x, QFC), (y, Bread) ]]
```

²Ace being a hardware store.

³now that webgrocer.com has gone bye-bye.

As usual, we'll break the problem into a number of steps. One point to bear in mind is that the particular constants and predicates like `Home`, `QFC` and `Sells` used in the discussion above are just examples; your code should work for any input of this sort. (And if this all seems a little daunting, rest assured that the solution is significantly shorter than this write-up!)

In ML, we will express predicates as

```
datatype atom = AtomConst of string | AtomVar of string
type predicate = { pred : string, vals : atom list }
```

A state is then `predicate list` (where the predicates better be over constant atoms), and a precondition is a `predicate expr`, like the following.

```
val state = [ {pred="At", vals=[AtomConst "home"]},
              {pred="Sells",vals=[AtomConst "QFC", AtomConst "Eggs"]} ];
val buy_precond = And({pred="At", vals=[AtomVar "place"]},
                     {pred="Sells", vals=[AtomVar "place",AtomVar "item"]});
```

In the problem descriptions below, we'll shorten statements like the above to `state=[At(home)]` and `buy_precond = And(At(place),Sells(place,item))`. See [Predicate Shorthand for ML](#) below to see how to make the shorthand actually work in ML.

1. The first thing to do is change variable atoms to constants. Write a Curried function `bindatom` that takes a pair of strings (x,c) and an atom `a`, and returns `AtomConst c` if `a` is `AtomVar x`, and otherwise returns `a` unchanged.
2. Write a function `bindpred (x,c) pred` that binds all atoms in `pred` according to the variable `x` and constant `c`. You *must* use `map` and `bindatom`. Note: see how writing `bindatom` as a Curried function makes things easier?
3. Write a function `filter` (from scratch) that takes a function `pred` and a list a_1, a_2, \dots , and returns the list of all a_i for which `pred a_i` is true. Write functions `getconsts` and `getvars` that take `atom lists` and return a list of all constant or variable atoms. You must use your `filter`.

4. Let `p` be a predicate. We say that a predicate `q` *matches* `p` if it has the same name, and the corresponding atoms are either both the same constant, or the atom in `p` is variable.

Write a function `satisfying_matches` that takes a predicate `p` and a state, and produces a list of all value lists from predicates in the state matching `p`. For example,

```
satisfying_matches Sells(x,y) [Sells(QFC,Eggs),Sells(Ace,Nails),
                              At(Home),Sells(QFC,Milk)]
--> [{"QFC","Eggs"}, {"Ace","Nails"}, {"QFC","Milk"}]
satisfying_matches Sells(QFC,y) [Sells(QFC,Eggs),Sells(Ace,Nails),
                                  At(Home),Sells(QFC,Milk)]
--> [{"QFC","Eggs"}, {"QFC","Milk"}]
```

5. Now write a function `get_binding` that given two lists of atoms of the same length, returns the implied binding as a list of string pairs. The idea is that pairing $[x,y]$ with $[QFC,Eggs]$ should produce the list $[("x","QFC"), ("y","Eggs")]$; that is, `x` is bound to `QFC` and `y` is bound to `Eggs`. Similarly, pairing $[QFC,y]$ with $[QFC,Eggs]$ gives $[("y","Eggs")]$. Define

an exception `BadGetBinding` and raise it if the lists are not the same length or otherwise not matching (you can assume `get.binding` will only be run on the atom lists of satisfying matches).

6. What's the point of having a binding if we don't apply it? Write a function `apply_binding` that takes a binding as returned from `get.binding` and a list of predicates, and returns a list of predicates with the binding applied. For example:

```
apply_binding [{"x","QFC"}, {"y","Eggs"}]
             [Sells(x,y),Have(y),Have(z),Have(Nails)]
--> [Sells(QFC,Eggs),Have(Eggs),Have(z),Have(Nails)]
```

Use `map` and `foldr`; your implementation may *not* be recursive. Hint: a single binding gets mapped across all predicates; the list of bindings is folded into the predicate list.

7. Write a one-line function `bindings_for_one_pred` that, given a predicate and a state, returns a list of the bindings for each satisfying match to the predicate in the state. Use `map`; your function cannot be recursive. Hint: the Curried implementation of `get.binding` makes this elegant.
8. In a comment, explain the difference—and there should be a difference!—between running `bindings_for_one_pred (At(Home))` with the state `[At(Home)]` and the state `[At(Away)]` (here both `Home` and `Away` are constant atoms).
9. The previous function gives a list of all the bindings for a state that satisfy a single predicate. We now want the bindings that satisfy an entire list of predicates. Write a function `bindings_for_preds` that takes a predicate list and a state and does just that. Use `bindings_for_one_pred`, `map` and `@`. Here are a few examples.

```
bindings_for_preds [At(x),Sells(x,y)] [At(QFC),Sells(QFC,Eggs)]
--> [[(x,QFC),(y,Eggs)]]
bindings_for_preds [At(x),Sells(x,y)] [At(QFC),Sells(QFC,Eggs),
                                         Sells(QFC,Milk)]
--> [[(x,QFC),(y,Eggs)],[(x,QFC),(y,Milk)]]
bindings_for_preds [At(x),Sells(x,y)] [At(QFC),Sells(QFC,Eggs)
                                         At(Ace),Sells(Ace,Nails)]
--> [[(x,QFC),(y,Eggs)],[(x,Ace),(y,Nails)]]
```

10. Suppose we had a precondition for buying things that looked like `(not Broke) and At(place) and Sells(place,item)`. Write this precondition as a predicate `expr` and bind it to `p1` (i.e., `val p1 = ...`). In a comment, give the output of running `satisfying_assignments` on `p1` and explain what that tells us about a state that satisfies that precondition.
11. This function puts it all together. Write a function `satisfying_bindings` that takes a state and a precondition, and returns a list of all bindings (if any) for which the state satisfies the precondition. Hint: hopefully after answering the last question you see that for a state to satisfy a predicate, it must find a binding that's good for some assignment returned by `satisfying_assignments`. This can be done by concatenating⁴ the results of doing the following to each satisfying assignment `asgn` of the precondition. Take each binding that

⁴`foldl` is useful here...

satisfies all the true predicates in `asgn`, apply that binding to the false predicates in `asgn`, and `keep`⁵ the bindings if the state doesn't include any of those false bindings⁶. Phew! Think about this carefully and make sure you understand what's going on before implementing it.

It's essential for this assignment that you really understand what's going on. There's not that much code to write—about 150 lines, in my solution, including the set functions from the last homework and the multiple `bind` implementations—but most of those lines required careful thought.

Extra Credit

Here are some ideas for extra credit. If you have another idea, talk to us. Remember the course policy on extra credit: if you're only concerned about your GPA, it's not worth your time. We've rated these extra credit problems where more stars means a harder problem.

1. (★★) Write an expression parser that takes a string like "x and (y or true)" and returns the corresponding `string expr`. If you're ambitious, use the IO structure to interact with Read-Eval-Print loop (see [Project 1](#) from CSE341 in winter 2004).
2. (★) Let's formalize *actions*, which describe how to change state. An action is defined over a list of variables, and is a precondition with an *effect*. An effect is a list of *allow* and *deny* predicates. For example, the action `BUY(x,y)` has the precondition `At(x) and Sells(x,y)` and effect `{allow=Have(y), deny=nil}`. An action `GO(x,y)` might have precondition `At(x)` and effect `{allow=At(y), deny=At(x)}`, so that `GO(Home,QFC)` requires that `At(Home)` be true, and changes the state so that `At(Home)` is not present and `At(QFC)` is.

For this extra-credit, write a datatype for actions, a function that tests if an action can be applied to a state, and a function that applies the allow and deny lists of an action to a state.

3. (★★) A problem is defined by a list of actions, a starting state and a goal state, and produces a sequence of actions that goes from the start to the goal. In our omlette example, the actions could be

```
GO(x,y)          = precond At(x)
                  effect allow=[At(y)], deny=[At(x)]
BUY(item,place) = precond At(place) and Sells(place,item)
                  effect allow=[Have(item)], deny=[]
MAKE_OMLETTE    = precond At(Home) and Have(Eggs)
                  effect allow=[Have(Omlette)], deny=[]
```

The start and goal states would then be

```
start = At(Home), Sells(QFC,Eggs), Sells(QFC,Bread), Sells(Ace, Hammers)
goal  = Have(Omlette)
```

A *plan* is a series of actions beginning from the start state that results in a state containing the goal state. In our example, a plan might be `GO(Home,QFC)`, `BUY(QFC,Eggs)`, `GO(QFC,Home)`, `MAKE_OMLETTE`.

⁵Maybe using `filter`?

⁶`isect` from the previous assignment will work well.

Write a planner in ML. One idea is to use breadth-first search, generating all states reachable by applying valid actions from the start state and searching until the goal is found. A better solution would use iterative-deepening depth-first search to avoid the exponential-space problem of breadth-first search. The key to this problem is finding the right representation of a tree in ML. Don't try to avoid searching the same states multiple times, that will needlessly complicate your solution.

Predicate Shorthand for ML

The shorthand for predicates I've been using is handy; it saves typing a lot of brackets, quotation marks and atom constructors. Here's a way to use similar notation in ML.

```
val x = AtomVar "x"; val y = AtomVar "y";
val QFC = AtomConst "QFC"; val Ace = AtomConst "Ace";
val Eggs = AtomConst "Eggs"; val Nails = AtomConst "Nails";
fun At(x) = {pred="At", vals=[x]};
fun Sells(x,y) = {pred="Sells",vals=[x,y]};
```

Now you can type things like `bindings_for_preds [At(x),Sells(x,y)]` `[At(QFC),Sells(QFC,Eggs)]` directly into ML. This has the advantage of also catching any typos when you type a predicate. Use this for testing! Extend with your own predicates! I did...

Type Bindings

Your solution should generate the following bindings (or their synonyms).

```
datatype 'a expr
  = And of 'a expr * 'a expr
  | Const of bool
  | Not of 'a expr
  | Or of 'a expr * 'a expr
  | Var of 'a
exception UnboundVar
infix mem union isect
val mem = fn : 'a * 'a list -> bool
val newmem = fn : 'a * 'a list -> 'a list
val setof = fn : 'a list -> 'a list
val union = fn : 'a list * 'a list -> 'a list
val isect = fn : 'a list * 'a list -> 'a list
val free_vars = fn : 'a expr -> 'a list
val eval = fn : 'a expr -> bool
val bind1 = fn : 'a -> bool -> 'a expr -> 'a expr
val bind = fn : ('a -> 'b expr) -> 'a expr -> 'b expr
val bindvar = fn : 'a -> bool -> 'a expr -> 'a expr
val changevar = fn : 'a -> 'a -> 'a expr -> 'a expr
val satisfying_assignments = fn : 'a expr -> ('a * bool) list list
datatype atom = AtomConst of string | AtomVar of string
```

```

type predicate = {pred:string, vals:atom list}
val bindatom = fn : string * string -> atom -> atom
val bindpred = fn : string * string -> predicate -> predicate
val filter = fn : ('a -> bool) -> 'a list -> 'a list
val getconsts = fn : atom list -> atom list
val getvars = fn : atom list -> atom list
val satisfying_matches = fn : predicate -> predicate list -> atom list list
exception BadGetBinding
val get_binding = fn : atom list -> atom list -> (string * string) list
val apply_binding = fn
  : (string * string) list -> predicate list -> predicate list
val bindings_for_one_pred = fn
  : predicate -> predicate list -> (string * string) list list
val bindings_for_preds = fn
  : predicate list -> predicate list -> (string * string) list list
val satisfying_bindings = fn
  : predicate list -> predicate expr -> (string * string) list list

```