CSE 373 Lecture 19: Wrap-Up of Sorting

- What's on our platter today?
  - How fast can the fastest sorting algorithm be?
    - Lower bound on comparison-based sorting
  - Tricks to sort faster than the lower bound
  - External versus Internal Sorting
  - Practical comparisons of internal sorting algorithms
  - Summary of sorting
- Covered in Chapter 7 of the textbook

How fast can we sort?

- Heapsort, Mergesort, and Quicksort all run in O(N log N) best case running time
- Can we do any better?
- Can we believe Pat Swe (pronounced "Sway") from Swetown (formerly **S**oft**w**ar**e**ville), USA, who claims to have discovered an O(N log log N) sorting algorithm?

The Answer is No! (if using comparisons only)

- Recall our basic assumption: we can only compare two elements at a time – how does this limit the run time?
- Suppose you are given N elements
  - Assume no duplicates – any sorting algorithm must also work for this case
- How many possible orderings can you get?
  - Example: a, b, c (N = 3)

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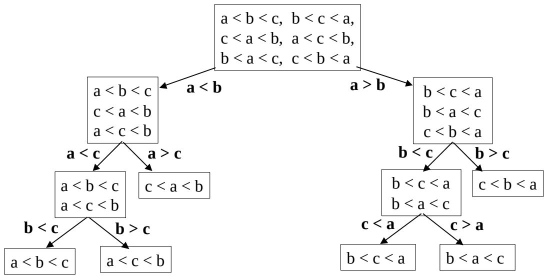
  - Example: a, b, c (N = 3)
  - Orderings: 1. a b c 2. b c a 3. c a b 4. a c b 5. b a c 6. c b a
  - 6 orderings = 3•2•1 = 3!
- For N elements: ___ ___ ___ … ___ ___ = N! orderings

N choices (N-1) choices 1 choice

A “Decision Tree”



a < b < c, b < c < a,
c < a < b, a < c < b,
b < a < c, c < b < a

a < b

a < b < c
c < a < b
a < c < b

a < c

a < b < c
a < c < b

b < c

a < b < c

b > c

a < c < b

a > c

c < a < b

a > b

b < c < a
b < a < c
c < b < a

b < c

b < c < a
b < a < c

c < a

b < c < a

c > a

b < a < c

b > c

c < b < a

Leaves contain possible orderings of a, b, c

Decision Trees and Sorting

- A Decision Tree is a Binary Tree such that:
  - Each node = a set of orderings
  - Each edge = 1 comparison
  - Each leaf = 1 unique ordering
  - How many leaves for N distinct elements?
- Only 1 leaf has sorted ordering
- Each sorting algorithm corresponds to a decision tree
  - Finds correct leaf by following edges (= comparisons)
- Run time ≥ maximum no. of comparisons
  - Depends on: <u>depth of decision tree</u>
  - <u>What is the depth of a decision tree for N distinct elements?</u>

Lower Bound on Comparison-Based Sorting

- Suppose you have a binary tree of <u>depth d</u> . How many leaves can the tree have?
  - E.g. depth d = 1 → at most 2 leaves, d = 2 → at most 4 leaves, etc.

Lower Bound on Comparison-Based Sorting

- A binary tree of <u>depth d</u> has <u>at most 2^d leaves</u>
  - E.g. depth d = 1 → 2 leaves, d = 2 → 4 leaves, etc.
  - Can prove by induction
- Number of leaves $L \leq 2^d$ → **$d \geq \log L$**
- Decision tree has L = N! leaves → its depth $d \geq \log(N!)$
  - What is log(N!)? (first, what is log(A•B)?)

Lower Bound on Comparison-Based Sorting

- Decision tree has L = N! leaves → its depth d ≥ log(N!)
  - What is log(N!)? (first, what is log(A•B)?)
  - log(N!) = log N + log(N-1) + … log(N/2) + … + log 1
    ≥ log N + log(N-1) + … log(N/2) (N/2 terms only)
    ≥ (N/2)•log(N/2) = **Ω(N log N)**
- Result: Any sorting algorithm based on comparisons between elements requires **Ω(N log N)** comparisons
  - Run time of any comparison-based sorting algorithm is **Ω(N log N)**
  - Can never get an O(N log log N) algorithm (sorry, Pat Swe!)

Hey! (you say)…what about Bucket Sort?

- Recall: Bucket sort → Elements are integers known to always be in the range 0 to B-1
  - Idea: Array Count has B slots ("*buckets*")
  1. Initialize: Count[i] = 0 for i = 0 to B-1
  2. Given input integer i, Count[i]++
  3. After reading all inputs, scan Count[i] for i = 0 to B-1 and print i if Count[i] is non-zero
- What is the running time for sorting N integers?

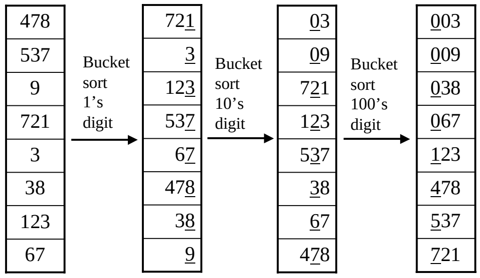
Hey! (you say)…what about Bucket Sort?

- Recall: Bucket sort → Elements are integers known to always be in the range 0 to B-1
  Idea: Array Count has B slots ("*buckets*")
  1. Initialize: Count[i] = 0 for i = 0 to B-1
  2. If input integer = i, Count[i]++
  3. After reading all inputs, scan Count[i] for i = 0 to B-1; print i if Count[i] ≠ 0 → sorted output
- What is the running time for sorting N integers?
  - Running Time: O(B+N) [B to zero/scan the array and N to read the input]
  - If B is Θ(N), then running time for Bucket sort = **O(N)**
  - **Doesn't this violate the O(N log N) lower bound result??**
- No – When we do Count[i]++, we are comparing one element with all B elements, not just two elements

Radix Sort = Stable Bucket Sort

- Problem: What if number of buckets needed is too large?
- Recall: Stable sort = a sort that does not change order of items with same key
- Radix sort = stable bucket sort on "slices" of key
  - E.g. Divide into integers/strings in digits/characters
  - Bucket-sort from least significant to most significant digit/character
  - Stability ensures keys already sorted stay sorted
  - Takes O(P(B+N)) time where P = number of digits

Radix Sort Example



478	721	03	003
537	3	09	009
9	123	721	038
721	537	123	067
3	67	537	123
38	478	38	478
123	38	67	537
67	9	478	721

Bucket sort 1's digit

Bucket sort 10's digit

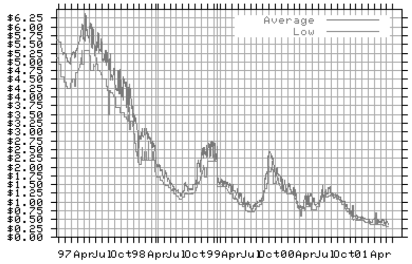
Bucket sort 100's digit

Internal versus External Sorting

- So far assumed that accessing A[i] is fast – Array A is stored in internal memory (RAM)
  - Algorithms so far are good for internal sorting
- What if A is so large that it doesn't fit in internal memory?
  - Data on disk or tape
  - Delay in accessing A[i] – e.g. need to spin disk and move head
- Need sorting algorithms that minimize disk/tape access time
  - External sorting – Basic Idea:
    - Load chunk of data into RAM, sort, store this "run" on disk/tape
    - Use the Merge routine from Mergesort to merge runs
    - Repeat until you have only one run (one sorted chunk)
    - Text gives some examples
- Waittaminute!! How important is external sorting?

Internal Memory is getting dirt cheap…

Price (in US$) for 1 MB of RAM

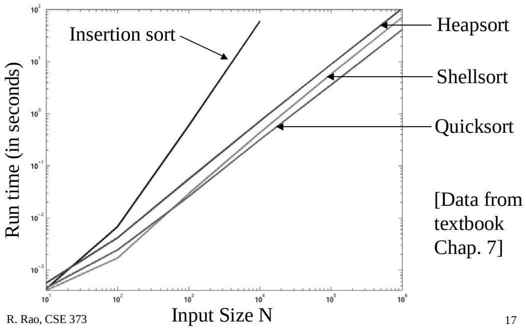


From: http://www.macresource.com/mrp/ramwatch/trend.shtml

External Sorting: A (soon-to-be) Relic of the Past?

- Price of internal memory is dropping, memory size is increasing, both at exponential rates (Moore's law)
- Quite likely that in the future, data will probably fit in internal memory for reasonably large input sizes
- Tapes seldom used these days – disks are faster and getting cheaper with greater capacity
- So, need not worry too much about external sorting
- For all practical purposes, internal sorting algorithms such as Quicksort should prove to be efficient

Okay…so let's talk about performance in practice



[Data from textbook Chap. 7]

Summary of Sorting

- Sorting choices:
  - $O(N^2)$ – Bubblesort, Selection Sort, Insertion Sort
  - $O(N^x)$ – Shellsort (x = 3/2, 4/3, 7/6, 2, etc. depending on increment sequence)
  - O(N log N) average case running time:
    - Heapsort: uses 2 comparisons to move data (between children and between child and parent) – may not be fast in practice (see graph)
    - Mergesort: easy to code but uses O(N) extra space
    - Quicksort: fastest in practice but trickier to code, $O(N^2)$ worst case
- Practical advice: When N is large, use Quicksort with median-of-three pivot. For small N (< 20), the N log N sorts are slower due to extra overhead (larger constants in big-oh notation)
  - For N < 20, use Insertion sort
  - E.g. In Quicksort, do insertion sort when sub-array size < 20 (instead of partitioning) and return this sorted sub-array for further processing

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Next time: Union-Find and Disjoint Sets

To do:

Finish reading chapter 7

Start reading chapter 8

Have a great weekend!