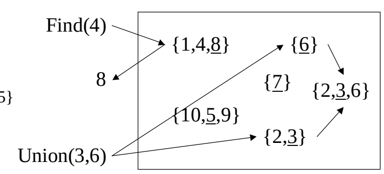
Lecture 21: Union and Find between Up-Trees

- Today's Agenda:
  - Planting and growing a forest of Up-Trees
    - Union-ing and Find-ing
    - Extended example
  - Implementing Union/Find
  - Smart Union and Find
    - Union-by-size/height and Path Compression
  - Run Time Analysis – as tough as it gets!
- Covered in Chapter 8 of the textbook

Recall from Last Time: Disjoint Set ADT

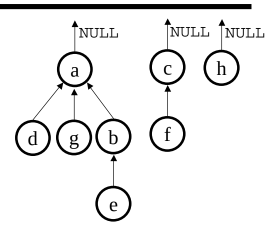
- Stores N unique elements. Two operations:
  - Find: Given an element, return the name of its equivalence class (its set)
  - Union: Given the names of two equivalence classes, merge them into one class

Example:
Initial Classes = {1,4,8}, {2,3}, {6}, {7}, {10,9,5}
Name of equiv. class underlined



Up-Tree Data Structure for Disjoint Sets

- Each equivalence class (or set) is an up-tree with its root as its representative member (= class name)
- All members of a given set are nodes in that set's up-tree
- Hash table maps input data to a node e.g. input string → integer index

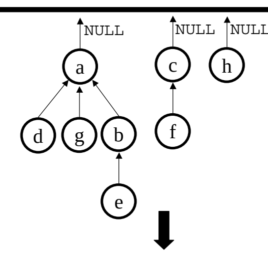


{a,d,g,b,e} {c,f} {h}

Up-trees are usually **not** binary!

Neat implementation trick for Up-Trees

- Forest of up-trees can easily be stored in an array (call it "up")
- If node names are integers or characters, can use a very simple, perfect hash function: Hash(X) = X
- `up[X]` = parent of X;
  = 0 if root



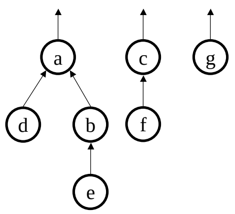
Array up:

0	1 (a)	2 (b)	3 (c)	4 (d)	5 (e)	6 (f)	7 (g)	8 (h)
-	**0**	1	**0**	1	2	3	1	**0**

Example of Find

Find: Just traverse to the root!

Find(f) = c
Find(e) = a



Runtime = ?

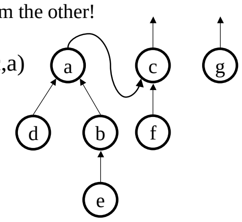
Array up:

0	1 (a)	2 (b)	3 (c)	4 (d)	5 (e)	6 (f)	7 (g)	8
-	**0**	1	**0**	1	2	3	**0**	

Example of Union

Union: Just hang one root from the other!

Union(c,a)



Runtime = ?

Now:
Find(f) = c
Find(e) = c

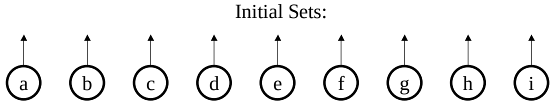
Array up:

0	1 (a)	2 (b)	3 (c)	4 (d)	5 (e)	6 (f)	7 (g)	8
-	**3**	1	**0**	1	2	3	**0**	

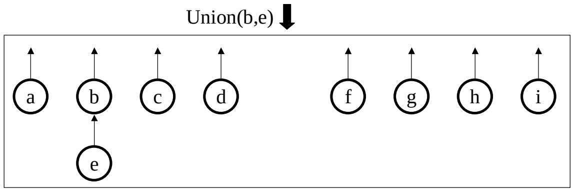
Change a (from 0) to point to c (= 3)

A more detailed example

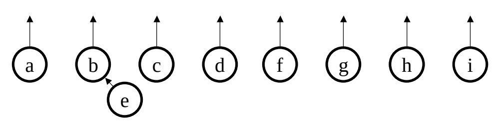
Initial Sets:



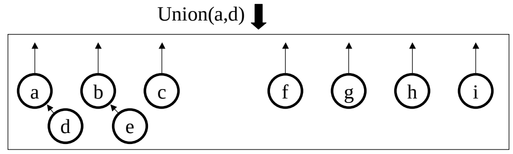
Union(b,e)



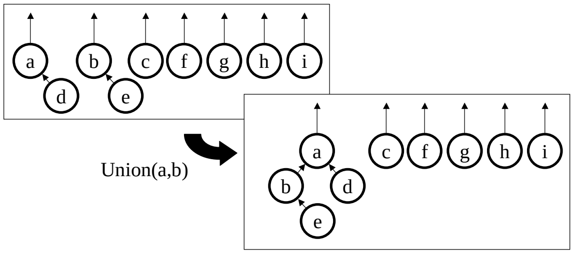
A more detailed example…



Union(a,d)

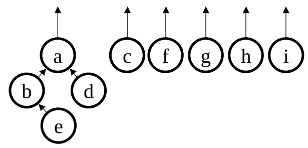


A more detailed example…



A more detailed example…

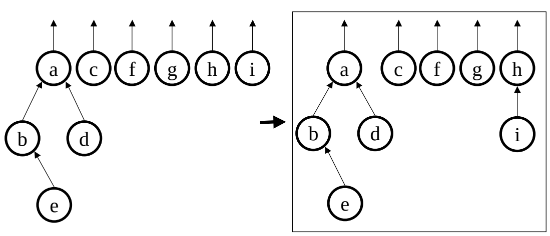
Union(d,e) – But (you say) d and e are not roots!
May be allowed in some implementations – do Find first to get roots
Since Find(d) = Find(e), union already done!



But: while we're finding **e**, could we do something to speed up Find(e) next time? (hold that thought!)

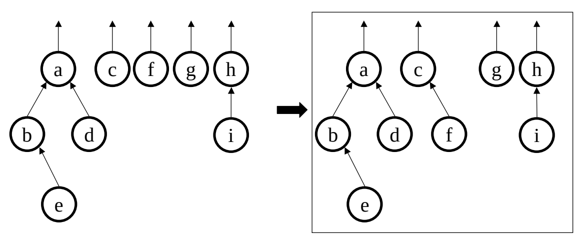
A more detailed example (continued)

Union(h,i)



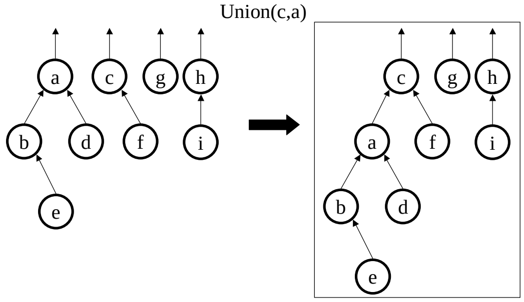
A more detailed example…

Union(c,f)



A more detailed example

Union(c,a)



Implementation of Find and Union

```
int Find(int X, DisjSet up)
{ // Assumes X = Hash(X_Element)
  // X_Element could be str/char etc.

  if (up[X] <= 0) // Root
   return X; //Return root = set name
  else
   //Find parent
   return Find(up[X], up);
}
```

```
void Union(DisjSet up,
   int X, int Y) {
   //Make sure X, Y are
   //roots
  assert(up[X] == 0);
  assert(up[Y] == 0);

  up[Y] = X;
 }
```

Runtime of Find: <u>O(max height)</u> Runtime of Union: <u>O(1)</u>

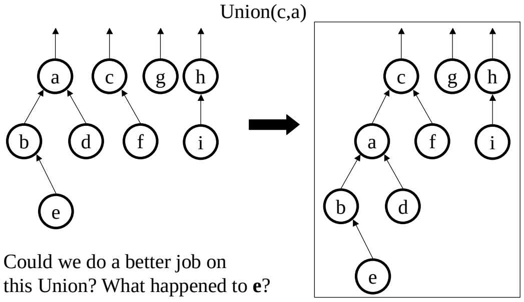
Height depends on previous Unions

→Best case: 1-2, 1-3, 1-4,… O(1)

→Worst case: 2-1, 3-2, 4-3,… <u>O(N)</u> Can we do better?

Let's look back at our example…

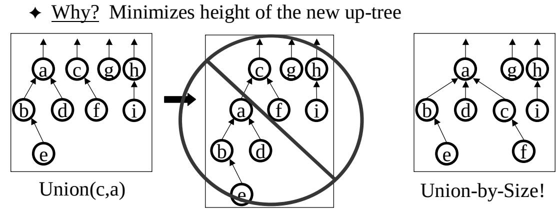
Union(c,a)



Could we do a better job on this Union? What happened to **e**?

Speeding Up Union/Find: Union-by-Size

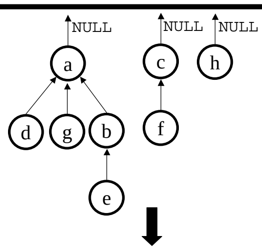
- For M Finds and N-1 Unions, worst case time is O(MN+N)
  - Can we speed things up by being clever about growing our up-trees?
- <u>Idea</u>: In Union, always make root of *larger* tree the new root
- <u>Why?</u> Minimizes height of the new up-tree



Union(c,a) Union-by-Size!

Trick for Storing Size Information

- Instead of storing 0 in root, store up-tree size as <u>negative value</u> in root node
- Why not positive value?
  - Would not know if array entry is size or parent pointer



Array up:

0	1 (a)	2 (b)	3 (c)	4 (d)	5 (e)	6 (f)	7 (g)	8 (h)
-	**-5**	1	**-2**	1	2	3	1	**-1**

Union-by-Size Code

```
void Union(DisjSet up, int X, int Y)
   {
    //X, Y are roots
    //containing (-size) of up-trees
  assert(up[X] < 0);
  assert(up[Y] < 0);

  if (-up[X] > -up[Y]) {
  //update size of X and root of Y
     up[X] += up[Y];
     up[Y] = X;
   }
  else { //size of X < size of Y
     up[Y] += up[X];
     up[X] = Y;
   }
}
```

New run time of Union = ?

New run time of Find = ?

Union-by-Size: Analysis

- Finds are O(max up-tree height) for a forest of up-trees containing N nodes
- Number of nodes in an up-tree of height h using union-by-size is $\geq 2^h$

> <u>Base case</u>: $h = 0$, tree has $2^0 = 1$ node
> <u>Induction hypothesis</u>: Assume true for $h < h'$
> <u>Induction Step</u>: New tree of height h' was formed via union of two trees of height h'-1
> Each tree then has $\geq 2^{h'-1}$ nodes by the induction hypothesis
> So, total nodes $\geq 2^{h'-1} + 2^{h'-1} = 2^{h'}$
> → True for all h

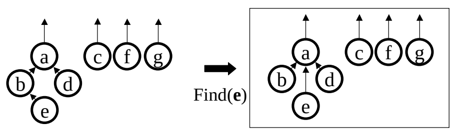
- Pick up-tree with max height
- Then, $2^{\text{max height}} \leq N$
- max height $\leq \log N$
- Find takes **O(log N)**

Union-by-Height

- Textbook describes alternative strategy of Union-by-height
- Keep track of height of each up-tree in the root nodes
- Union makes root of up-tree with greater height the new root
- Same results and similar implementation as Union-by-Size
  - Find is O(log N) and Union is O(1)

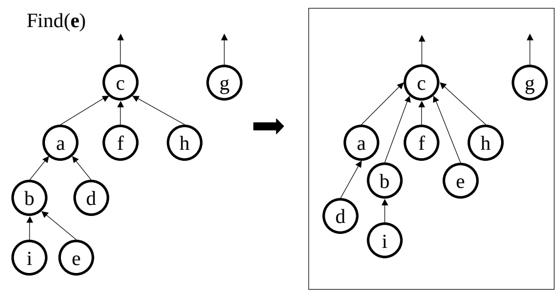
Speeding Up Find: Path Compression

- If we do M Finds on same element → O(M log N) time
  - Can we modify Find to have *side-effects* so that next Find will be faster?
- Path Compression: Point everything along path of a Find to root
- Reduces height of entire access path to 1: Finds get faster!
  - Déjà vu? Idea similar to the one behind your old friend – splay tree…



Path compression!

A P.C. example with more meat…



How to P.C. – Path Compression Code

```
int Find(int X, DisjSet up)
{ // Assumes X = Hash(X_Element)
  // X_Element could be str/char etc.

  if (up[X] <= 0) // Root
   return X; //Return root = set name
  else
   //Find parent
   return up[X] = Find(up[X], up);
}
```

Make all nodes along access path point to root

- Trivial modification of original Find
- New running time of Find = ?

How to P.C. – Path Compression Code

```
int Find(int X, DisjSet up)
{ // Assumes X = Hash(X_Element)
  // X_Element could be str/char etc.

  if (up[X] <= 0) // Root
   return X; //Return root = set name
  else
   //Find parent
   return up[X] = Find(up[X], up);
}
```

Collapsing the tree by pointing to root

- Find still takes O(max up-tree height) worst case
- But what happens to the tree heights over time?
- What is the *amortized* run time of Find if we do M Finds?

Analysis of P.C. with Union-by-Size

- R. E. Tarjan (of the up-trees fame) showed that:
  - When both P.C. and Union-by-Size are used, the worst case run time for a sequence of M operations (Unions or Finds) is $\Theta(M\ \alpha(M,N))$
- What is $\alpha(M,N)$?
  - $\alpha(M,N)$ is the inverse of Ackermann's function
- What is Ackermann's function?

Digression: Them slow-growing functions…

- How fast does log N grow? log N = 4 for N = 16 = 2^4
  - Grows quite slowly
- Let $\log^{(k)}$ N = log (log (log … (log N))) (*k* logs)
- Let log* N = minimum *k* such that $\log^{(k)}$ N ≤ 1
- How fast does $\log^*$ N grow? $\log^*$ N = 4 for N = 65536 = $2^{2^{2^2}}$
  - Grows very slowly
- Ackermann created a really explosive function A(i, j) whose inverse α(M, N) grows very, very slowly (slower than $\log^*$ N)
- How slow does α(M, N) grow? α(M, N) = 4 for M (≥ N) **far** larger than the number of atoms in the universe (2^{300})!!

Analysis of P.C. with Union-by-Size

- R. E. Tarjan (of the up-trees fame) showed that:
  - When both P.C. and Union-by-Size are used, the worst case run time for a sequence of M operations (Unions or Finds) is $\Theta(M\ \alpha(M,N))$
  - $\alpha(M, N) \leq 4$ for all practical choices of M and N
- Textbook proves weaker result of O(M log* N) time
  - 7 pages and 8 Lemmas! (Check it out but no need to know the proof)
- Amortized run time per operation = total time/(# operations) = $\Theta(M\ \alpha(M,N))/M = \Theta(\alpha(M,N)) \approx \Theta(1)$ for all practical purposes (constant time!)

Summary of Disjoint Set and Union/Find

- Disjoint Set data structure arises in many applications where objects of interest fall into different equivalence classes or sets
  - Cities on a map, electrical components on chip, computers in a network, people related to each other by blood, etc.
- Two main operations: Union of two classes and Find class name for a given element
- Up-Tree data structure allows efficient array implementation
  - Unions take O(1) worst case time, Finds can take O(N)
  - Union-by-Size reduces worst case time for Find to O(log N)
  - Union-by-Size plus Path Compression allows further speedup
    - Any sequence of M Union/Find operations results in O(1) amortized time per operation (for all practical purposes)

Next Class: CSE 373 gets graphic…

(Algo-rhythms on Graphs)

To Do:

Finish Homework #4 (due next class)

Finish reading chapter 8

Start reading chapter 9