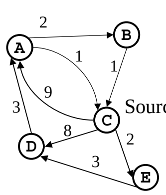
Lecture 24: From Dijkstra to Prim

- Today's Topics:
  - Dijkstra's Shortest Path Algorithm
  - Depth First Search
  - Spanning Trees
  - Minimum Spanning Trees
    - Prim's Algorithm
- Covered in Chapter 9 in the textbook

Single Source, Shortest Path Problem

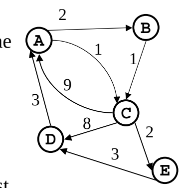
- Given a graph G = (V, E) and a "source" vertex s in V, find the minimum cost paths from s to every vertex in V



Source

Dijkstra's Shortest Path Algorithm

1. Initialize the cost of each node to ∞
2. Initialize the cost of the source to 0
3. While there are unknown nodes left in the graph
   1. Select the unknown node N with the *lowest cost* (greedy choice)
   2. Mark N as known
   3. For each node A adjacent to N
      If (N's cost + cost of (N, A)) < A's cost
         A's cost = N's cost + cost of (N, A)
         Prev[A] = N *//store preceding node*



(Prev allows paths to be reconstructed)

Dijkstra's Algorithm (greed in action)

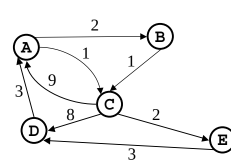
vertex	known	cost	Prev
A	No	∞	-
B	No	∞	-
C	Yes	0	-
D	No	∞	-
E	No	∞	-

Initial

→

vertex	known	cost	Prev
A	Yes	8	D
B	Yes	10	A
C	Yes	0	-
D	Yes	5	E
E	Yes	2	C

Final



Analysis of Dijkstra's Algorithm

- Main loop:
  While there are unknown nodes left in the graph ← $|V|$ times
  1. Select the unknown node N with the *lowest cost* ← $O(|V|)$
  2. Mark N as known
  3. For each node A adjacent to N ← $O(|E|)$ total
     If (N's cost + cost of (N, A)) < A's cost
     A's cost = N's cost + cost of (N, A)

Total time = $|V|\,(O(|V|)) + O(|E|) = O(|V|^2 + |E|)$
Dense graph: $|E| = \Theta(|V|^2) \rightarrow$ Total time = $O(|V|^2) = O(|E|)$ $\surd$
Sparse graph: $|E| = \Theta(|V|) \rightarrow$ Total time = $O(|V|^2) = O(|E|^2)$ χ
Quadratic! Can we do better?

Analysis of Dijkstra's Algorithm

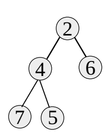
Yes! Use a priority queue to store vertices with key = cost

$|V|$ times:
Select the unknown node N with the *lowest cost*

→ deleteMin

$|E|$ times:
A's cost = N's cost + cost of (N, A)

→ decreaseKey



Total run time = ?

Analysis of Dijkstra's Algorithm

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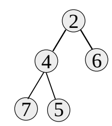
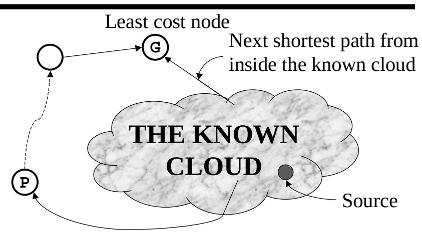


Total run time = $O(|V| \log |V| + |E| \log |V|)$

Does Dijkstra's Algorithm Always Work?

- Dijkstra's algorithm is an example of a greedy algorithm
- Greedy algorithms always make choices that currently seem the best
  - Short-sighted – no consideration of long-term or global issues
  - Locally optimal does not always mean globally optimal
- In Dijkstra's case – choose the least cost node, but what if there is another path through other vertices that is cheaper?
- Can prove: Never happens if all edge weights are positive

The “Cloudy” Proof of Dijkstra’s Correctness



If the path to **G** is the next shortest path, the path to **P** must be at least as long.

Therefore, any path through **P** to **G** cannot be shorter!

Inside the Cloud (Proof)

Everything inside the cloud has the correct shortest path

Proof is <u>by induction</u> on the # of nodes in the cloud:

- Base case: Initial cloud is just the source with shortest path 0
- Inductive hypothesis: cloud of k-1 nodes all have shortest paths
- Inductive step: choose the least cost node G → has to be the shortest path to G (previous slide). Add k^{th} node G to the cloud

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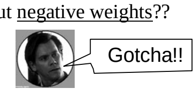
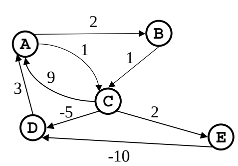
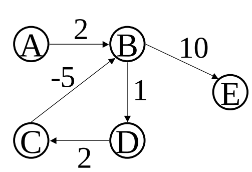
But waitaminute!! What about <u>negative weights</u>??



Negative Weights: Dijkstra’s Achilles Heel



<u>Dijkstra</u>: C→D (cost = -5)
Least cost path:
C→E→D (cost = -8)



<u>Negative cycles</u>: What’s the shortest path from A to E? (or to B, C, or D, for that matter)

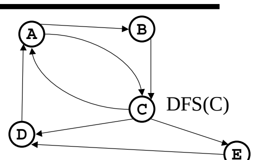
Depth First Search (DFS)

- We used Breadth First Search for finding shortest paths in an unweighted graph
  - Use a queue to explore neighbors of source vertex, neighbors of each neighbor, and so on: 1 edge away, two edges away, etc.
- Its counterpart: Depth First Search
  - A second way to explore all nodes in a graph
- DFS searches down one path as deep as possible
  - When no new nodes available, it *backtracks*
  - When backtracking, we explore side-paths that weren't taken
- DFS allows an easy recursive implementation
  - So, DFS uses a stack while BFS uses a queue

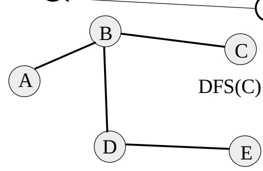
DFS Pseudocode

- Pseudocode for DFS:

```
DFS(v)
If v is unvisited
  mark v as visited
  print v (or process v)
  for each edge (v,w)
   DFS(w)
```



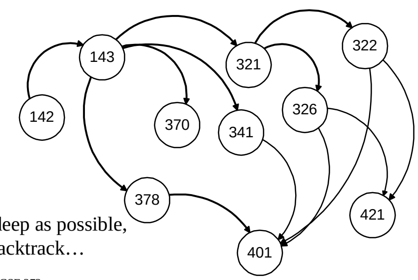
- Works for directed or undirected graphs



- Running time = **O(|*V*| + |*E*|)**

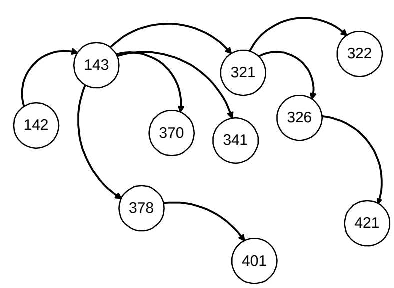
What about DFS on this graph?

- What happens when you do DFS("142")?

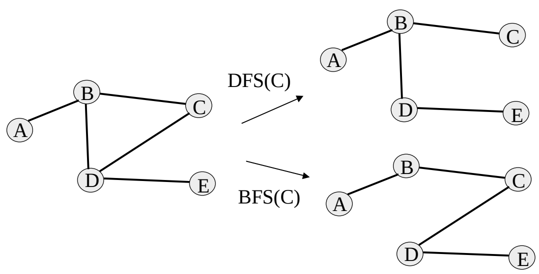


Go as deep as possible,
Then backtrack…

We get a "spanning" tree…

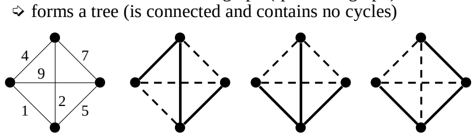


DFS and BFS may give different trees…



Spanning Tree Definition

- *Spanning tree*: a subset of edges from a connected graph that:
  - touches all vertices in the graph (*spans* the graph)
  - forms a tree (is connected and contains no cycles)

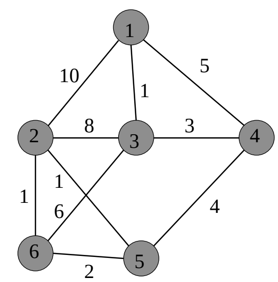


- *Minimum spanning tree*: the spanning tree with the least total edge cost

Minimum Spanning Tree (MST)

We are given a weighted, undirected graph $G = (V, E)$, with weight function $w: E \rightarrow \mathbf{R}$ mapping edges to real valued weights

<u>Problem</u>: Find the minimum cost spanning tree

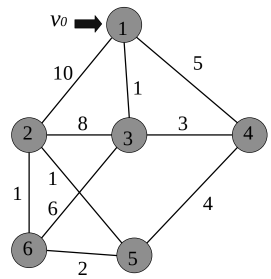


Why minimum spanning trees?

- Lots of applications
- Minimize length of gas pipelines between cities
- Find cheapest way to wire a house (with minimum cable)
- Find a way to connect various routers on a network that minimizes total delay
- Etc…

Prim's Algorithm for Finding the MST

1. Starting from an empty tree, *T*, pick a vertex, *v0*, at random and initialize: *V'* = {*v0*} and *E'* = {}



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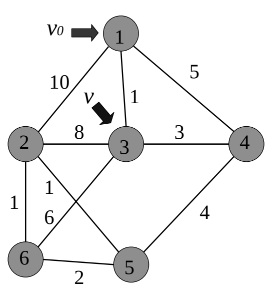

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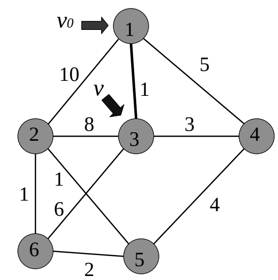

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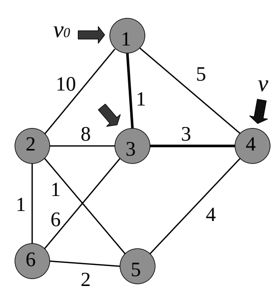
4. Repeat until all vertices have been added



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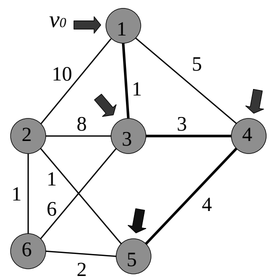

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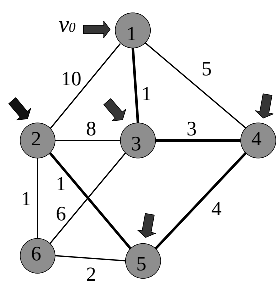

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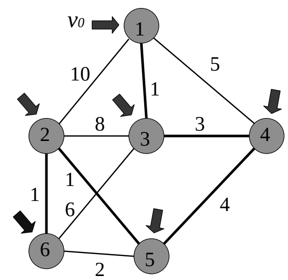
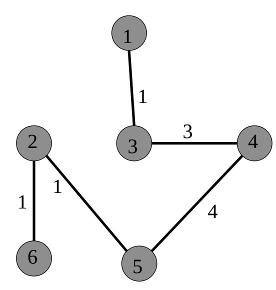


Prim's Algorithm for Finding the MST

Done!
Total cost = 1 + 3 + 4 + 1 + 1
= 10



Next Class:

Analysis of Prim's Algorithm

Kruskal takes a bow – faster MST

To Do:

Programming Assignment #2

(Don't wait until the last few days!!!)

Continue chapter 9