Shortest Paths

CSE 373

Data Structures

Lecture 21

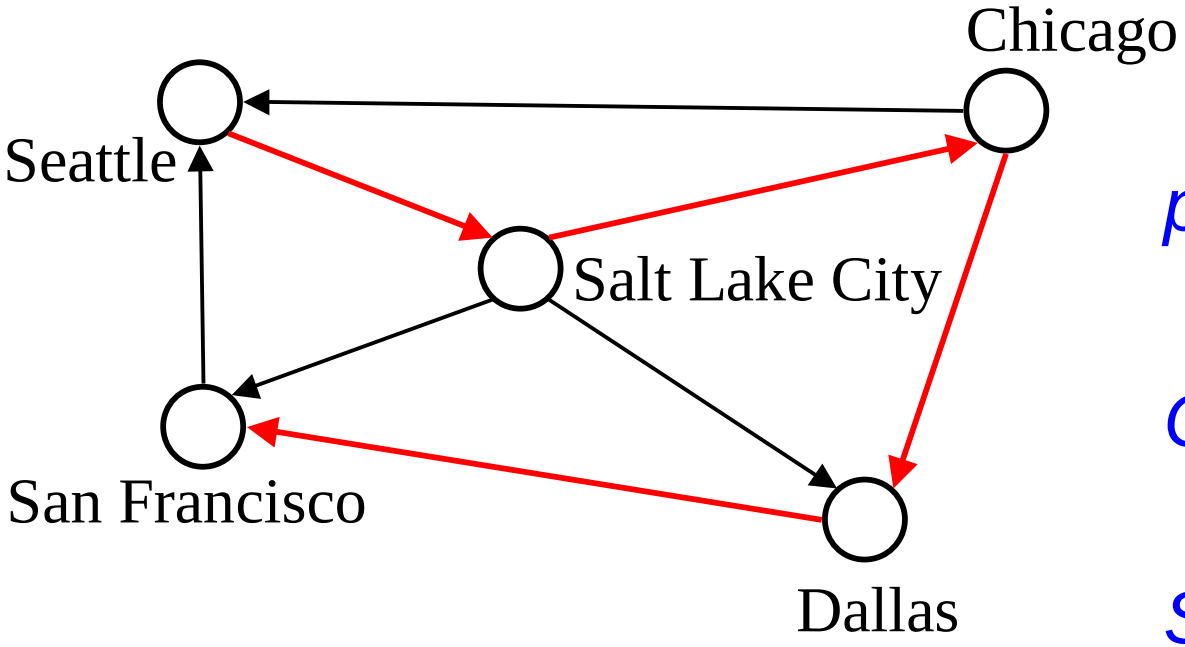
Readings and References

- Reading
  - Section 9.3 , Section 10.3.4

Some slides based on: CSE 326 by S. Wolfman, 2000

Path

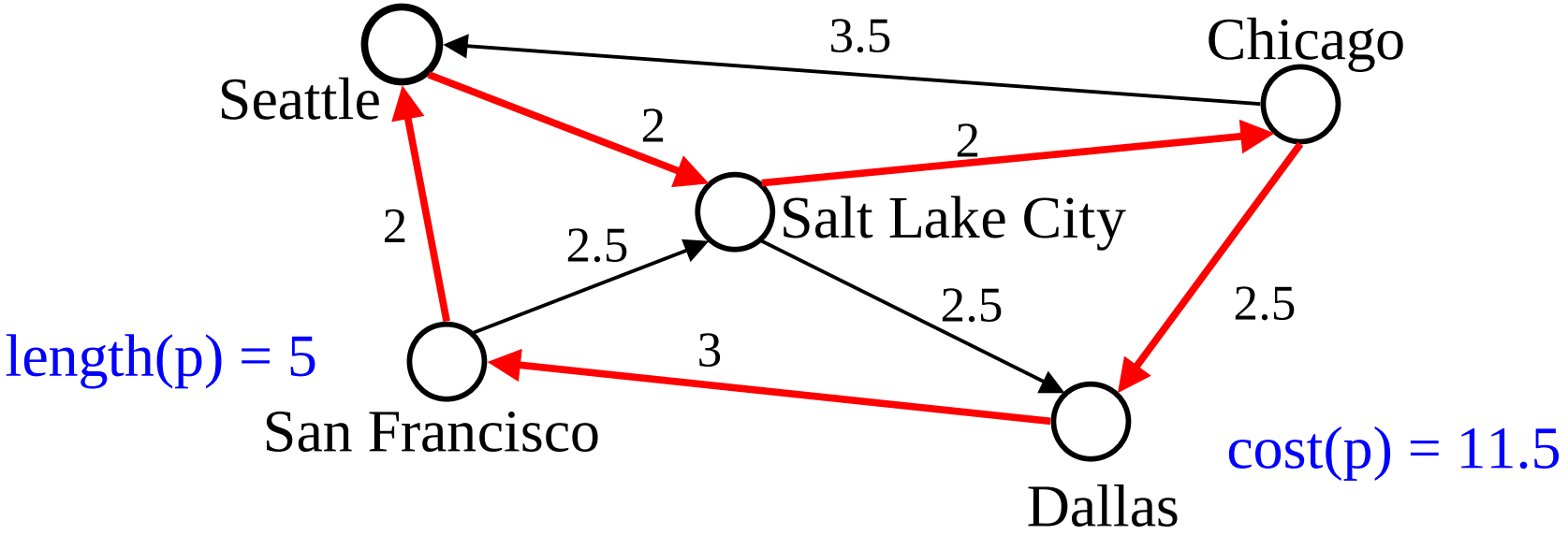
- A *path* is a list of vertices $\{v_1, v_2, \ldots, v_n\}$ such that (v_i, v_{i+1}) is in **E** for all $0 \leq i < n$.



p = {Seattle, Salt Lake City, Chicago, Dallas, San Francisco}

Path cost and Path length

- *Path cost*: the sum of the costs of each edge
- *Path length*: the number of edges in the path
  - Path length is the unweighted path cost (each edge = 1)



Shortest Path Problems

- Given a graph G = (*V*, *E*) and a "source" vertex *s* in *V*, find the minimum cost paths from *s* to every vertex in *V*
- Many variations:
  - unweighted vs. weighted
  - cyclic vs. acyclic
  - pos. weights only vs. pos. and neg. weights
  - etc

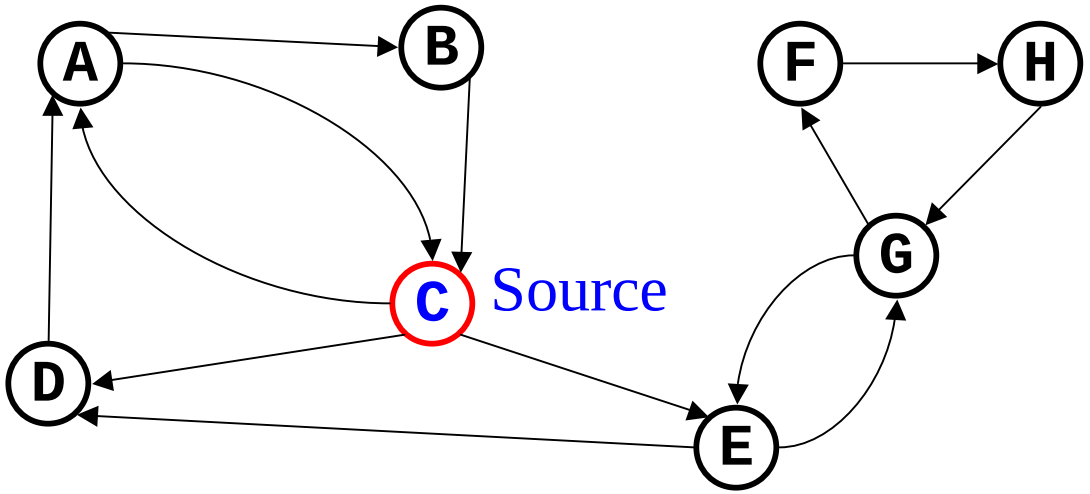
Why study shortest path problems?

- Traveling on a budget: What is the cheapest airline schedule from Seattle to city X?
- Optimizing routing of packets on the internet:
  - Vertices are routers and edges are network links with different delays. What is the routing path with smallest total delay?
- Shipping: Find which highways and roads to take to minimize total delay due to traffic

Unweighted Shortest Path Problem

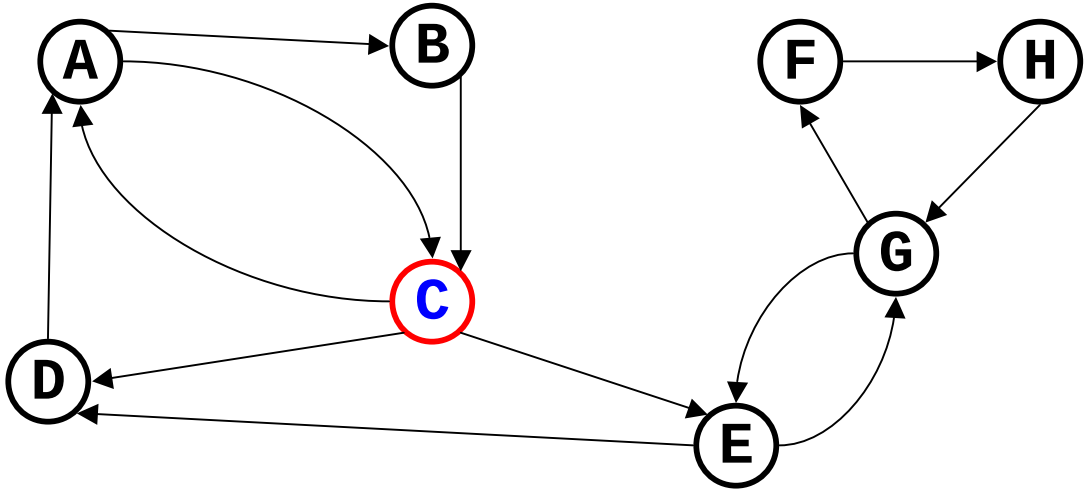
Problem: Given a "source" vertex *s* in an unweighted graph

G = (*V*,*E*), find the shortest path from *s* to all vertices in G



Breadth-First Search Solution

- Basic Idea: Starting at node s, find vertices that can be reached using 0, 1, 2, 3, …, N-1 edges (works even for cyclic graphs!)



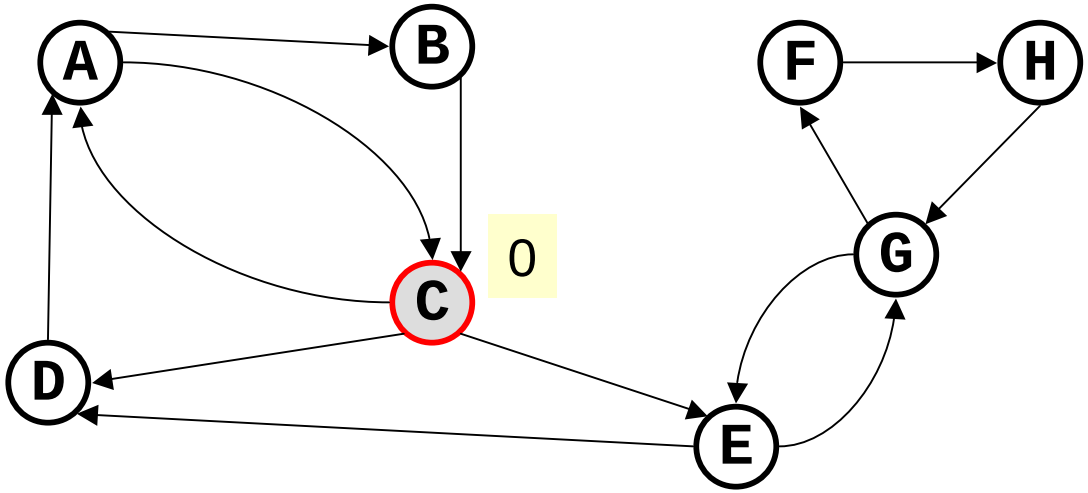
Breadth-First Search Algorithm

- Uses a queue to track vertices that are “nearby”
- source vertex is **s**

```
Distance[s] := 0
Enqueue(Q,s); Mark(s)
while queue is not empty do
    X := Dequeue(Q);
    for each vertex Y adjacent to X do
       if Y is unmarked then
          Distance[Y] := Distance[X] + 1;
          Previous[Y] := X;
          Enqueue(Q,Y); Mark(Y);
```

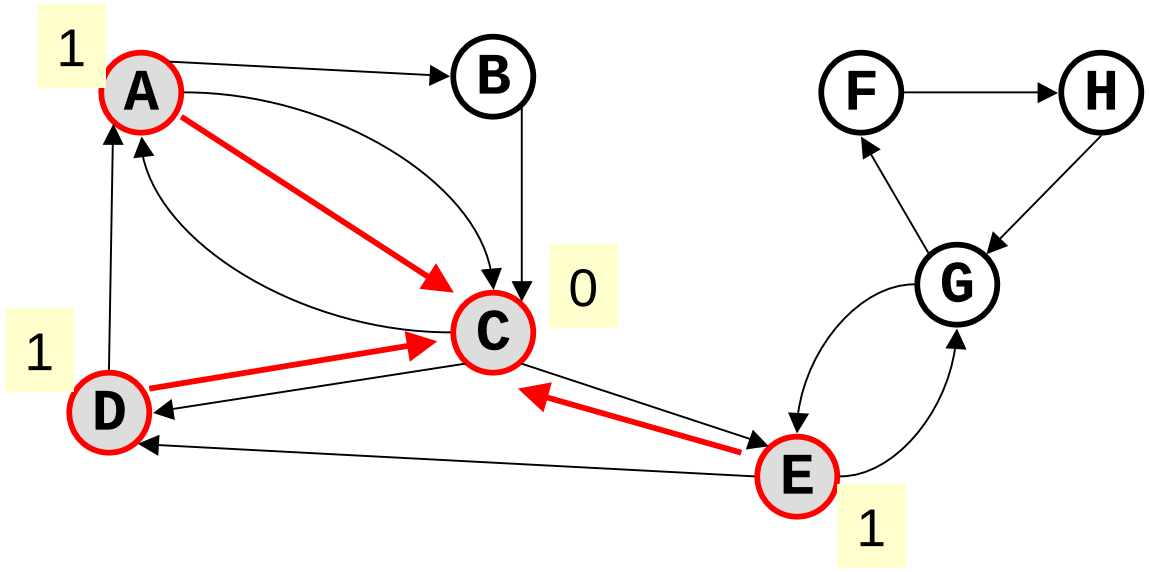
- Running time = $O(|V| + |E|)$

Shortest Path



Q = C

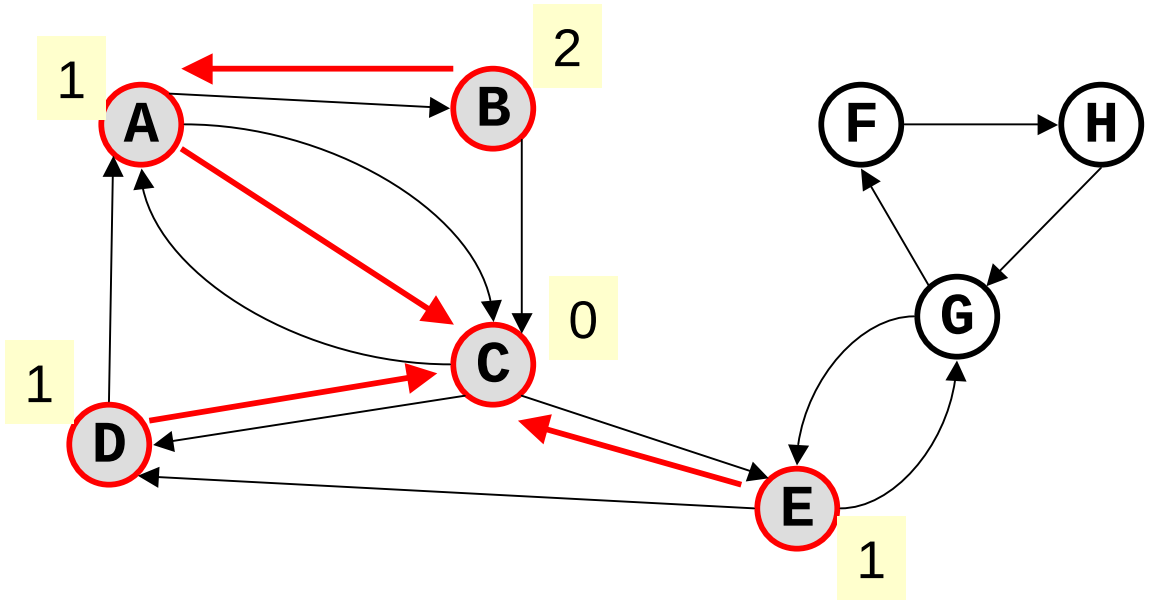
Shortest Path



Q = A D E

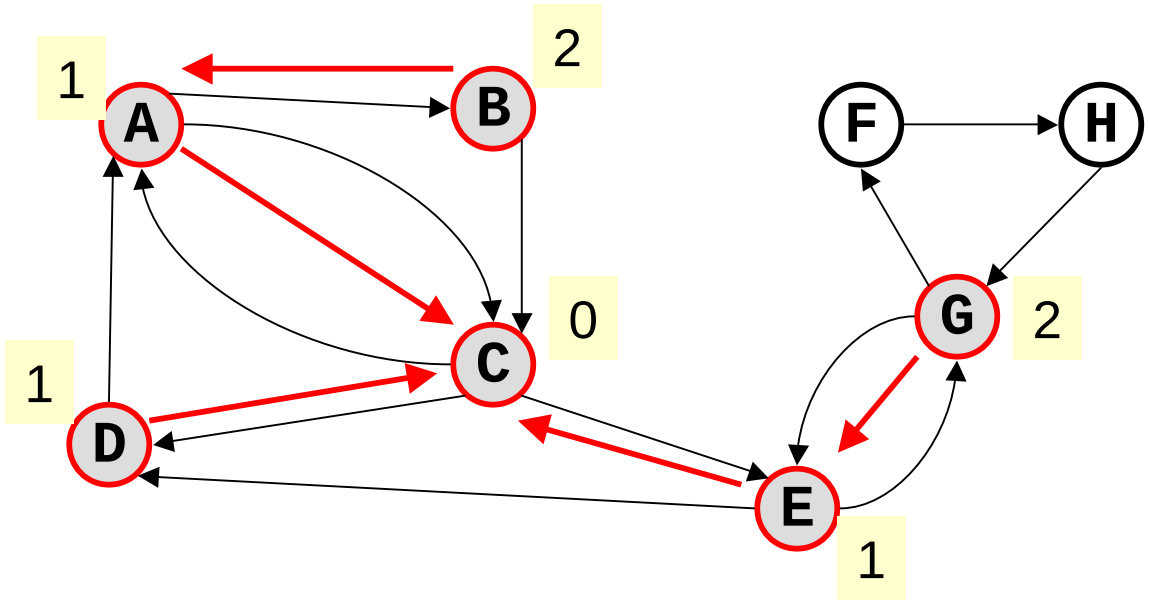
Previous pointer

Shortest Path



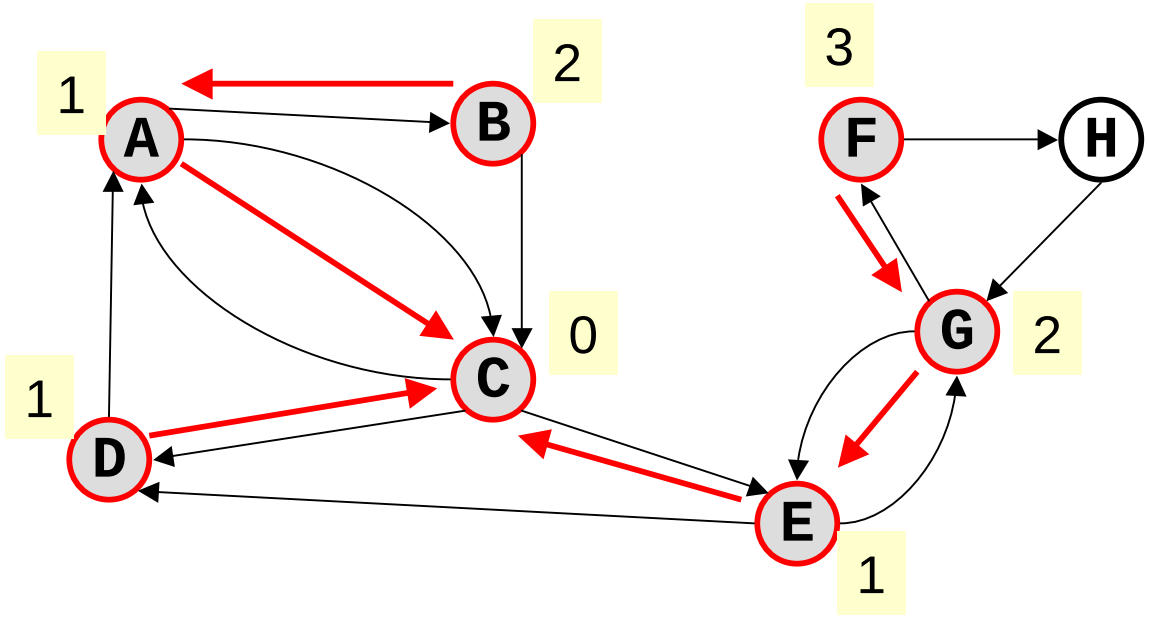
Q = D E B

Shortest Path



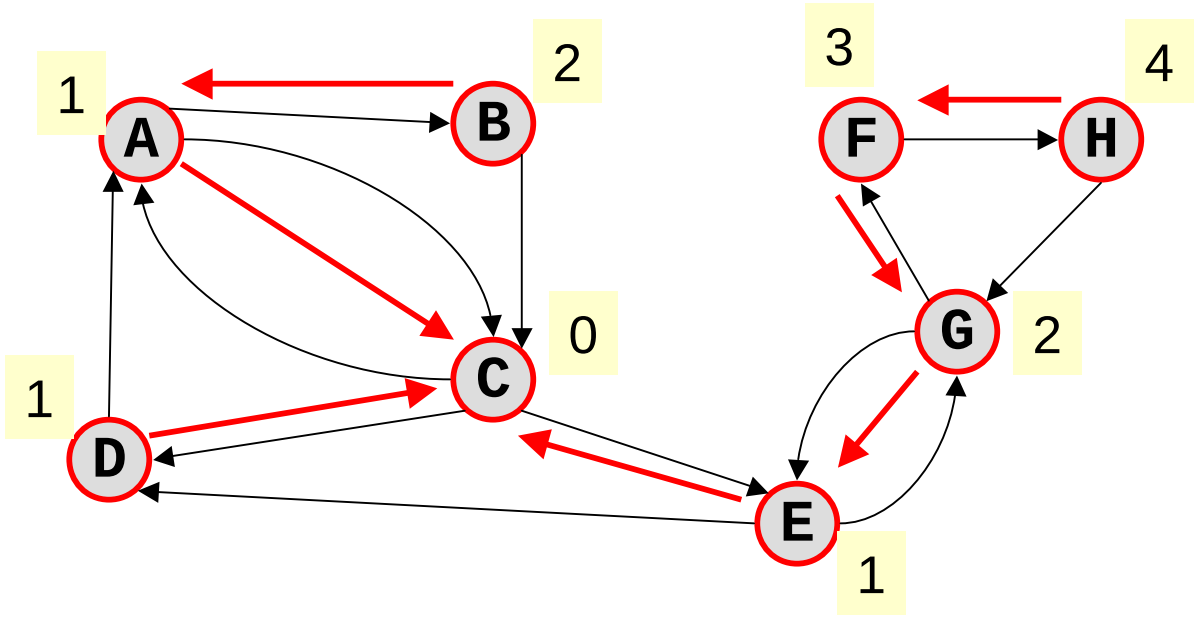
Q = B G

Shortest Path



Q = F

Shortest Path



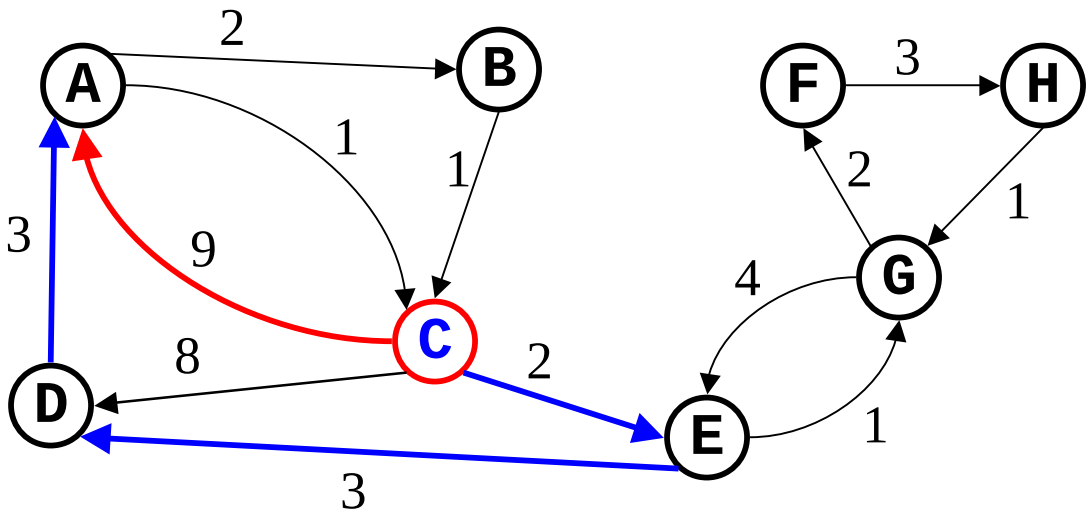
Q = H

What if edges have weights?

- Breadth First Search does not work anymore
  - › minimum *cost* path may have more edges than minimum *length* path

Shortest path from C to A:
Cà A (cost = 9)

Minimum Cost Path = Cà Eà Dà A (cost = 8)



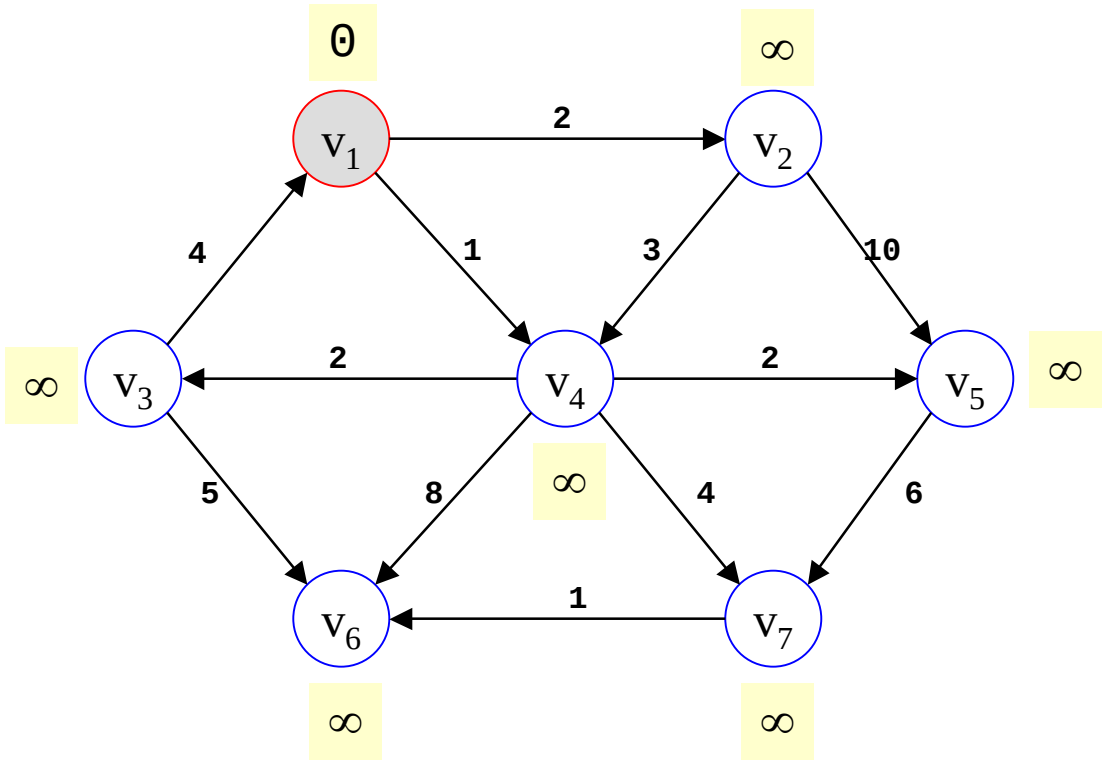
Dijkstra's Algorithm for Weighted Shortest Path

- Classic algorithm for solving shortest path in weighted graphs (without negative weights)
- A greedy algorithm (irrevocably makes decisions without considering future consequences)
- Each vertex has a cost for path from initial vertex

Dijkstra's Shortest Path Algorithm

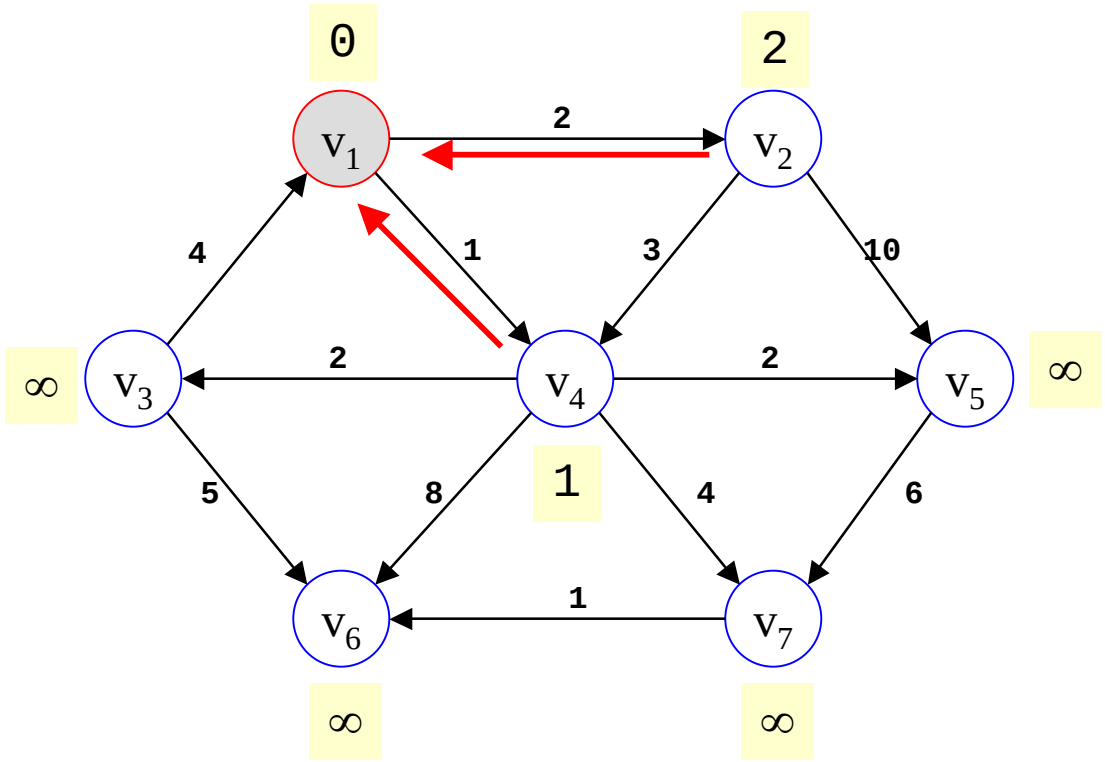
- Initialize the cost of s to 0, and all the rest of the nodes to ∞
- Initialize set S to be $\varnothing$
  - S is the set of nodes to which we have a shortest path
- While S is not all vertices
  - Select the node A with the lowest cost that is not in S and identify the node as now being in S
  - for each node B adjacent to A
    - if cost(A)+cost(A,B) < B's currently known cost
      - set cost(B) = cost(A)+cost(A,B)
      - set previous(B) = A so that we can remember the path

A weighted directed graph



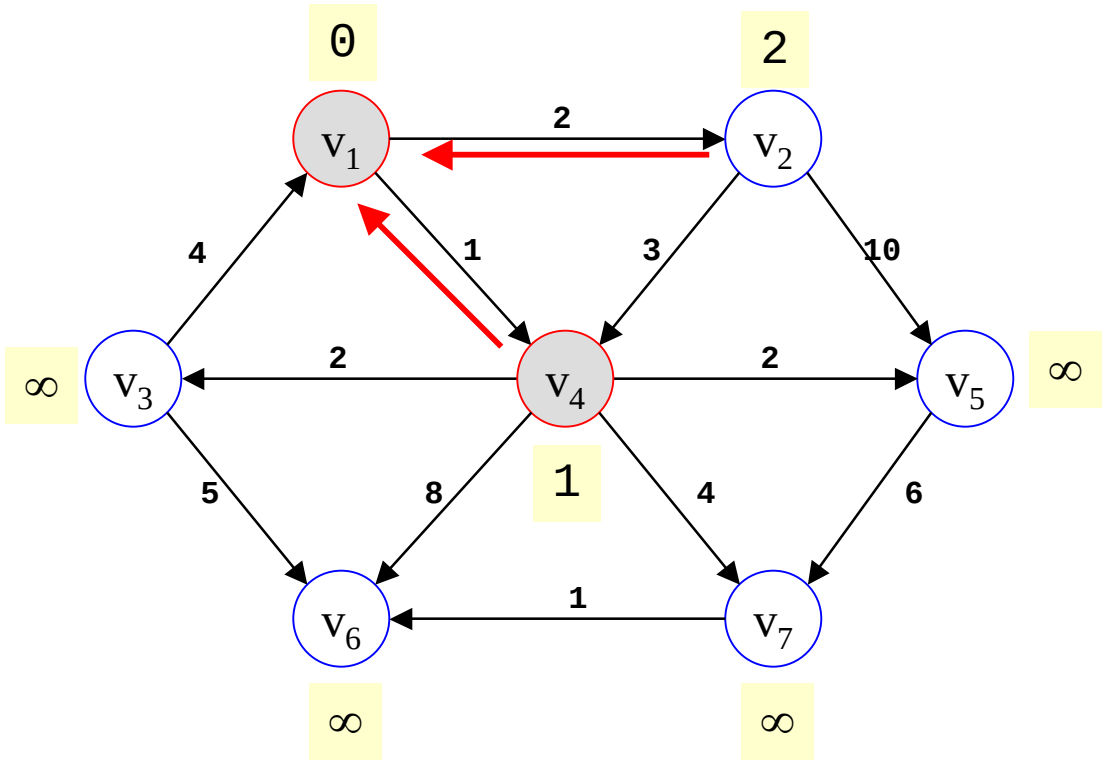
Pick vertex not in S with lowest cost.

A weighted directed graph



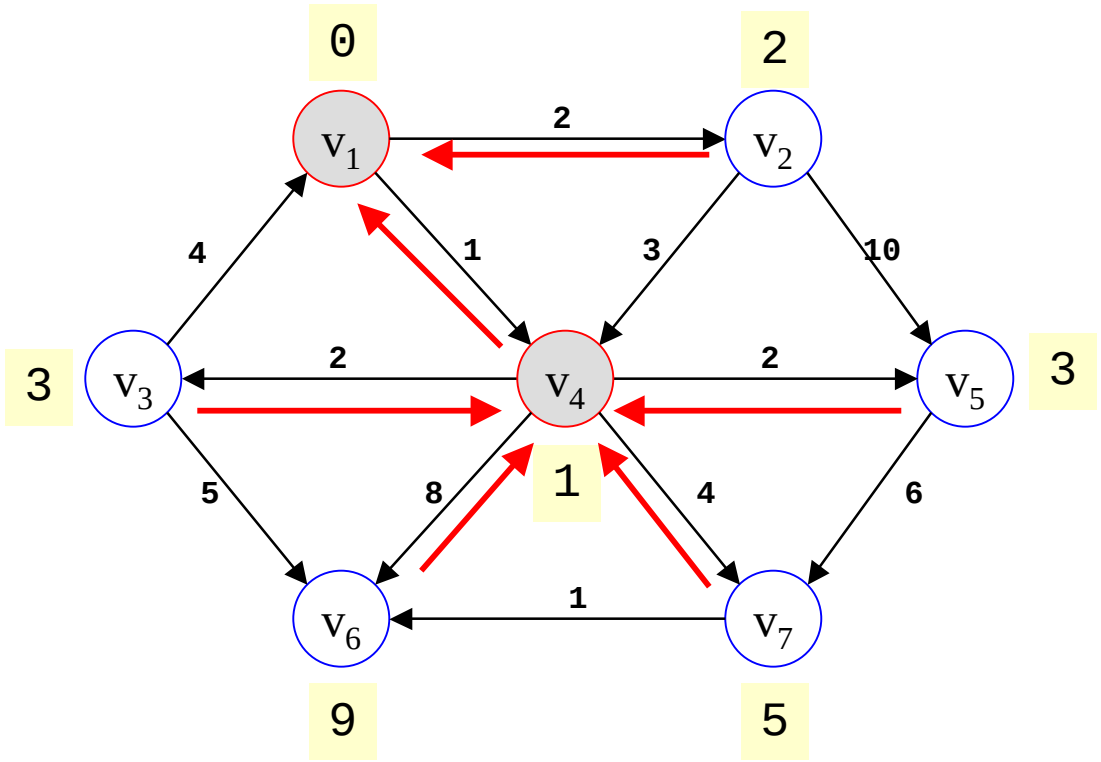
Update neighbors

A weighted directed graph



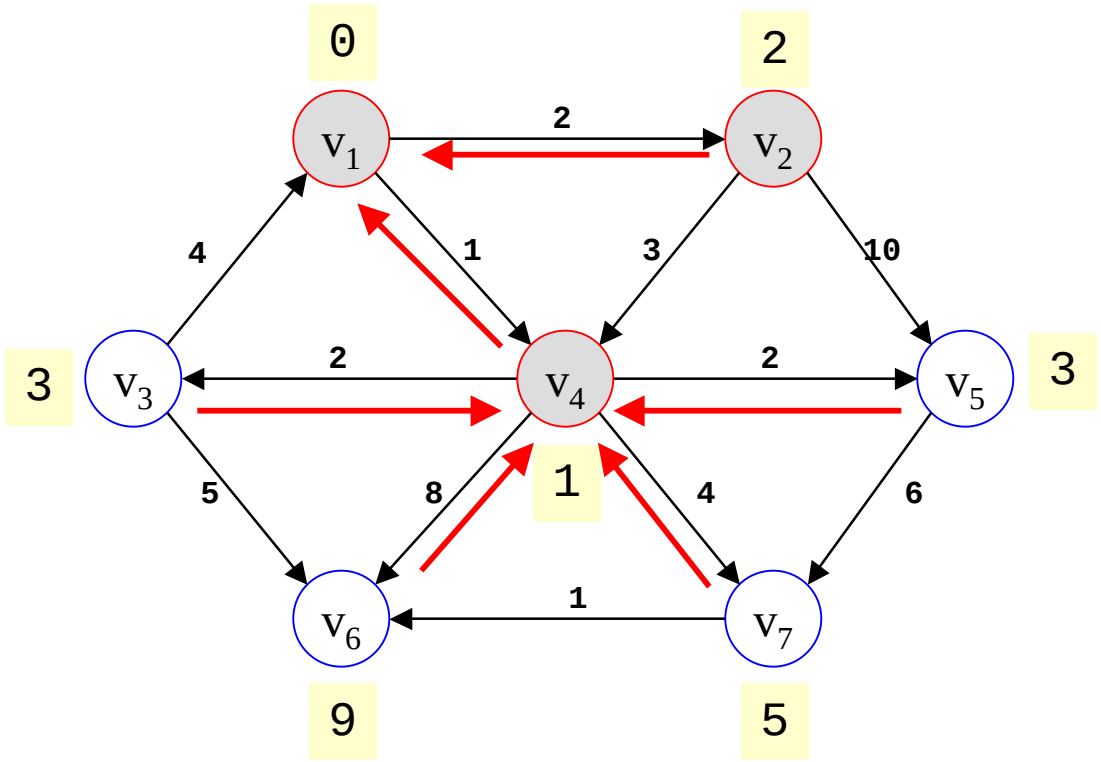
Pick vertex not in S with lowest cost

A weighted directed graph



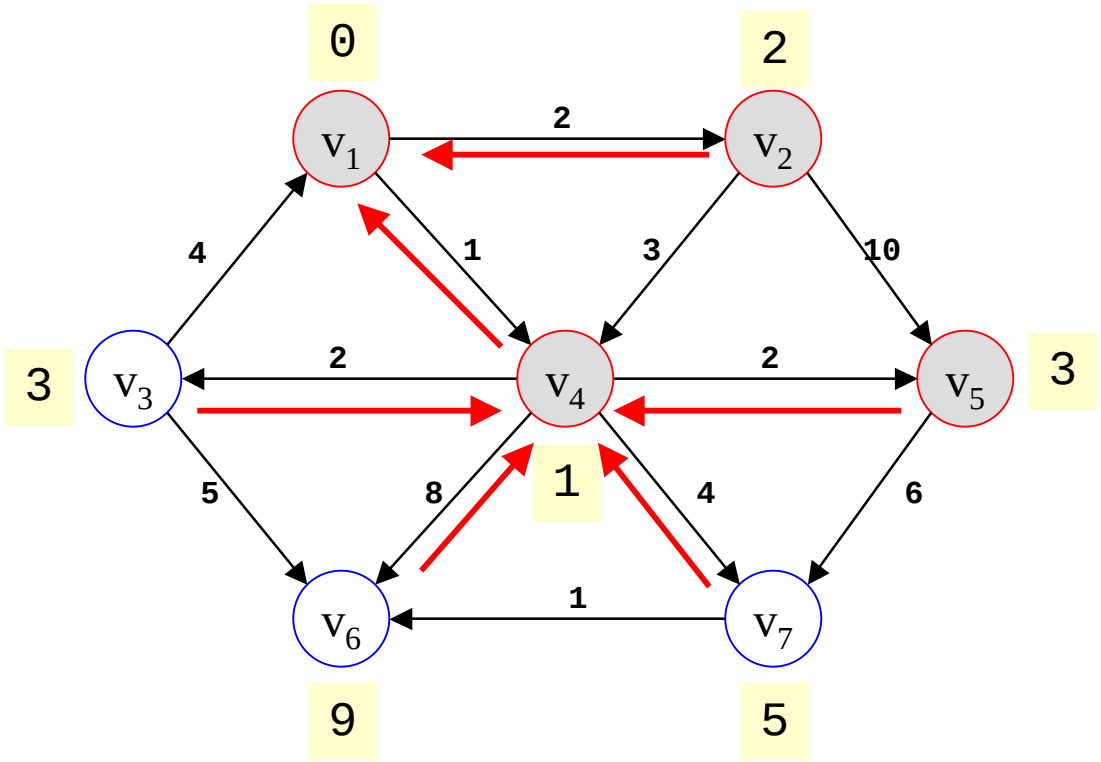
Update neighbors

A weighted directed graph



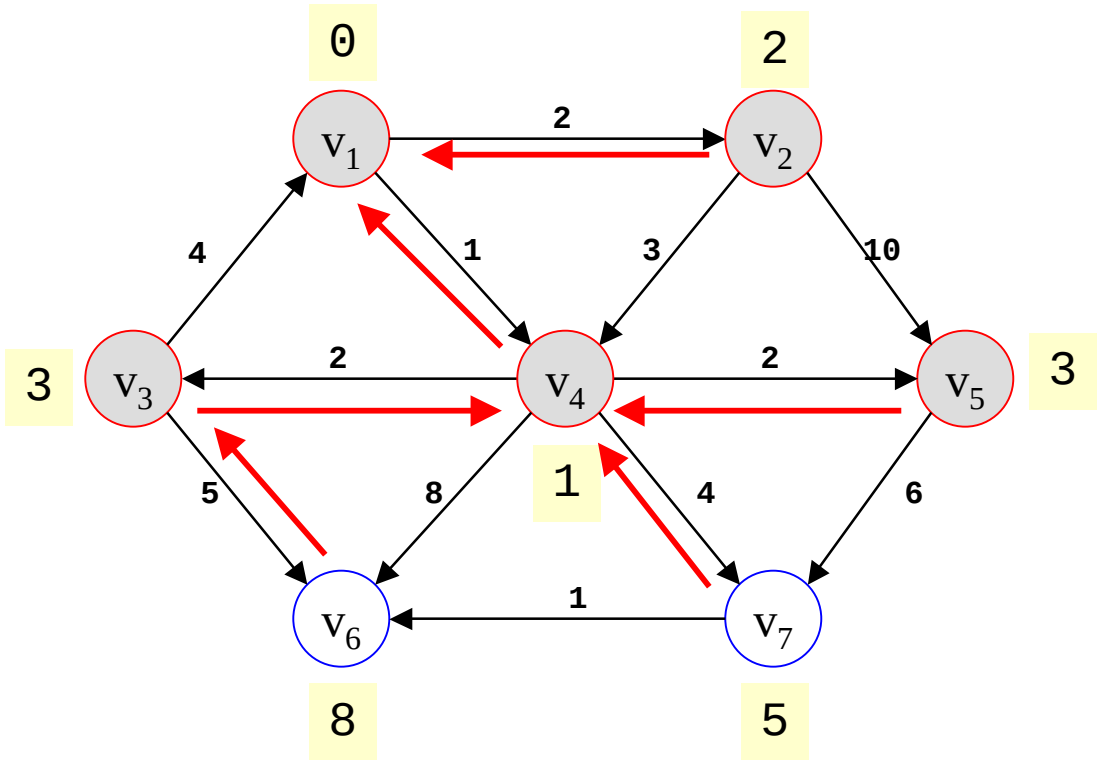
Pick vertex not in S with lowest cost and update neighbors

A weighted directed graph



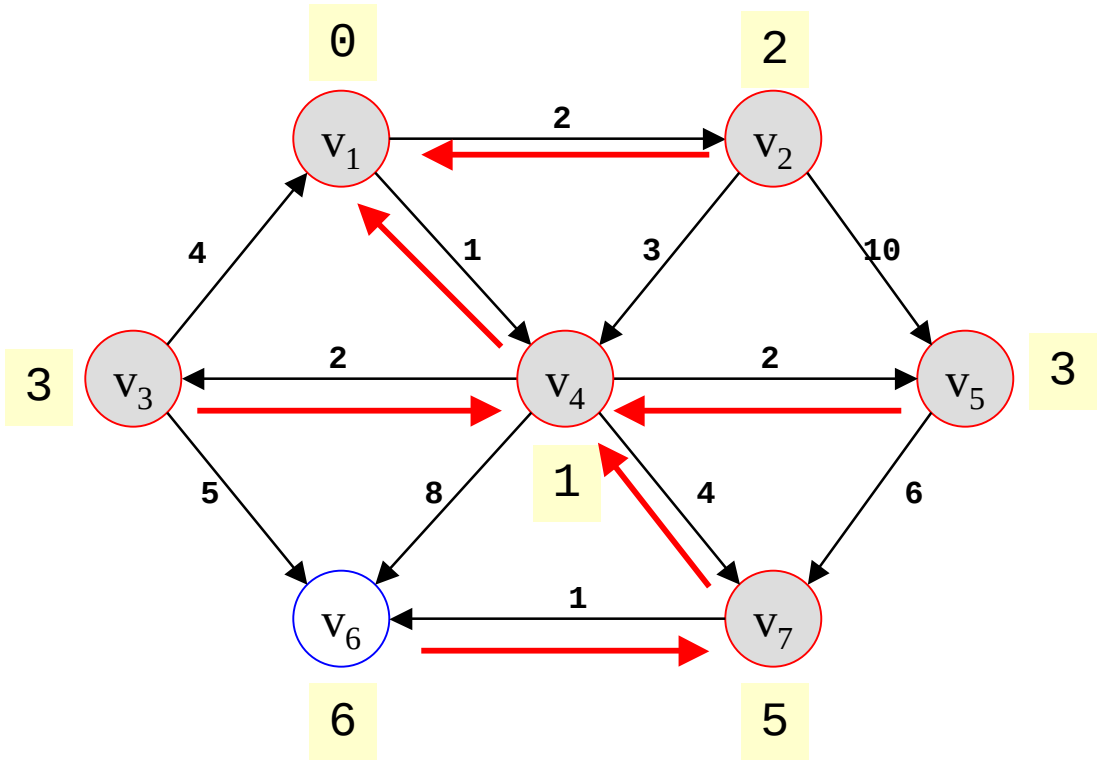
Pick vertex not in S with lowest cost and update neighbors

A weighted directed graph



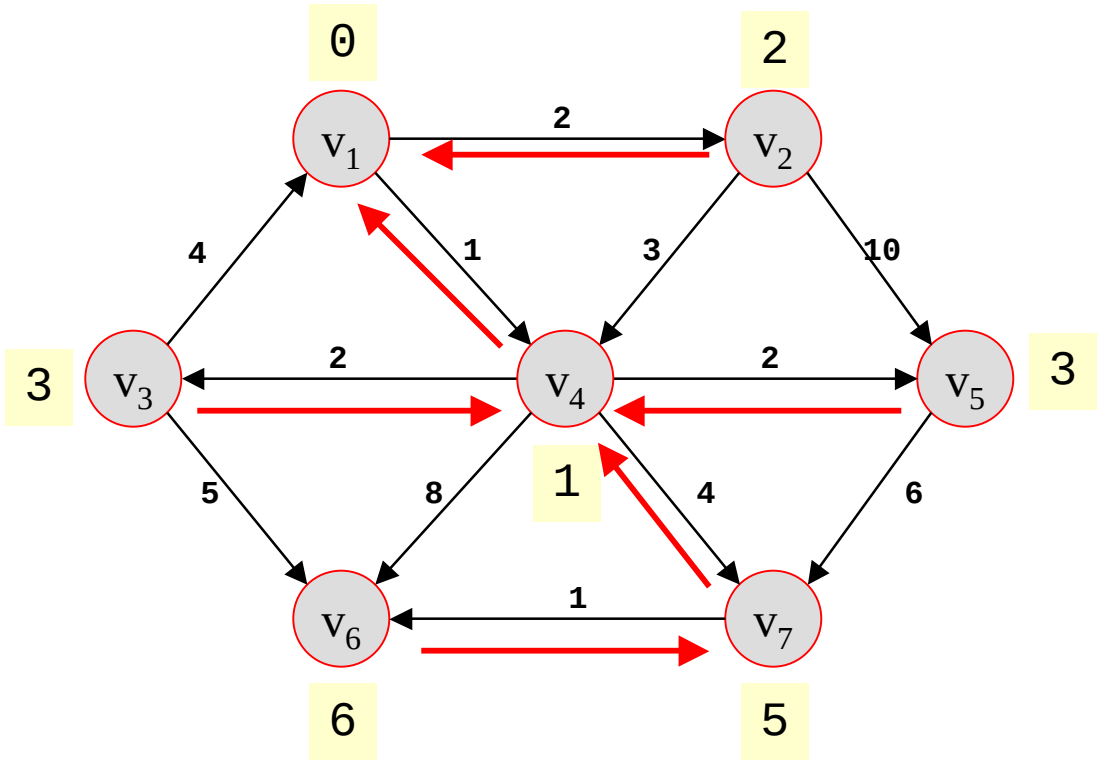
Pick vertex not in S with lowest cost and update neighbors

A weighted directed graph



Pick vertex not in S with lowest cost and update neighbors

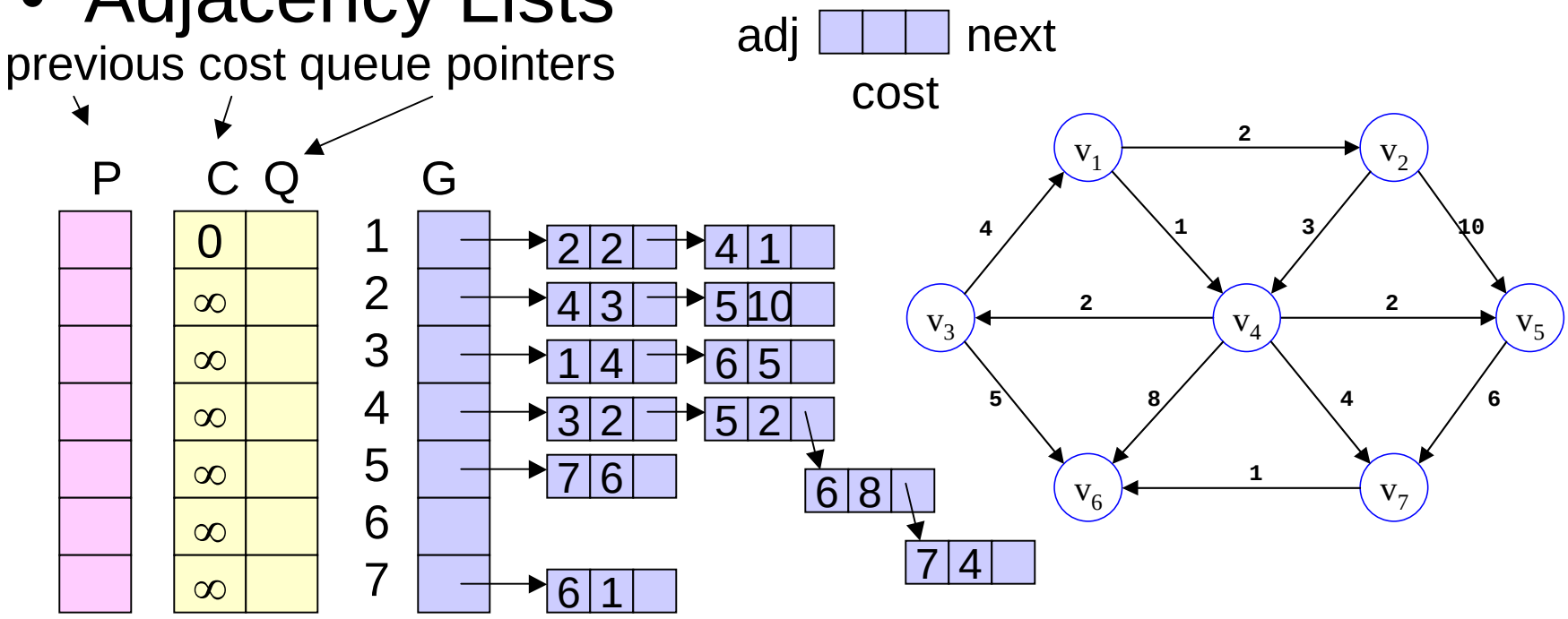
A weighted directed graph



Pick vertex not in S with lowest cost and update neighbors

Data Structures

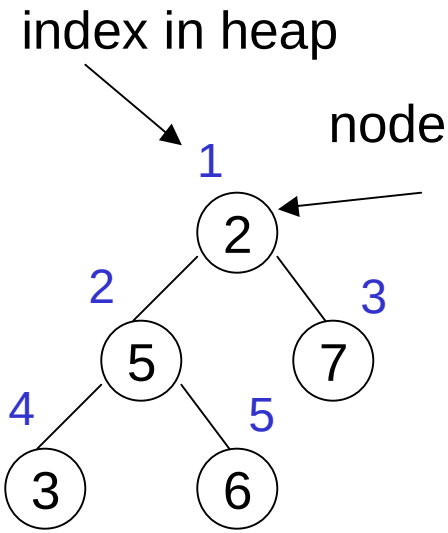
- Adjacency Lists

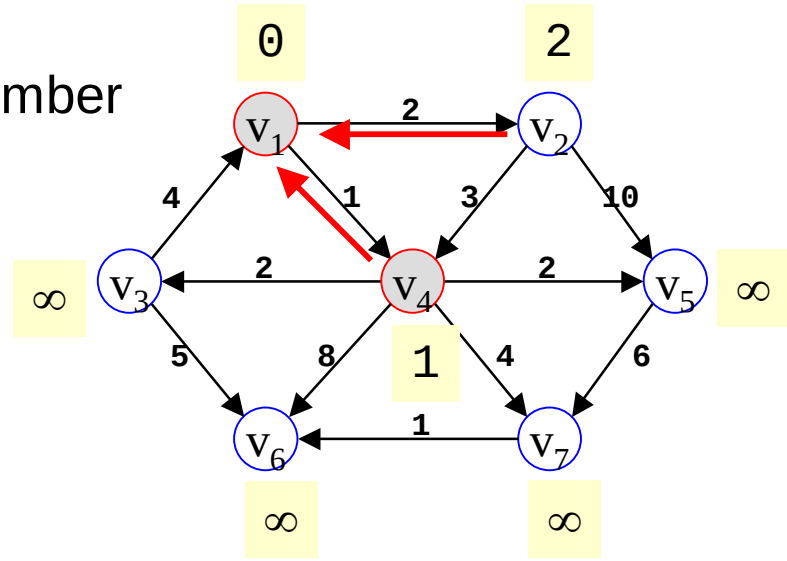


Priority queue for finding finding and deleting lowest cost vertex and for decreasing costs (Binary Heap works)

Priority Queue

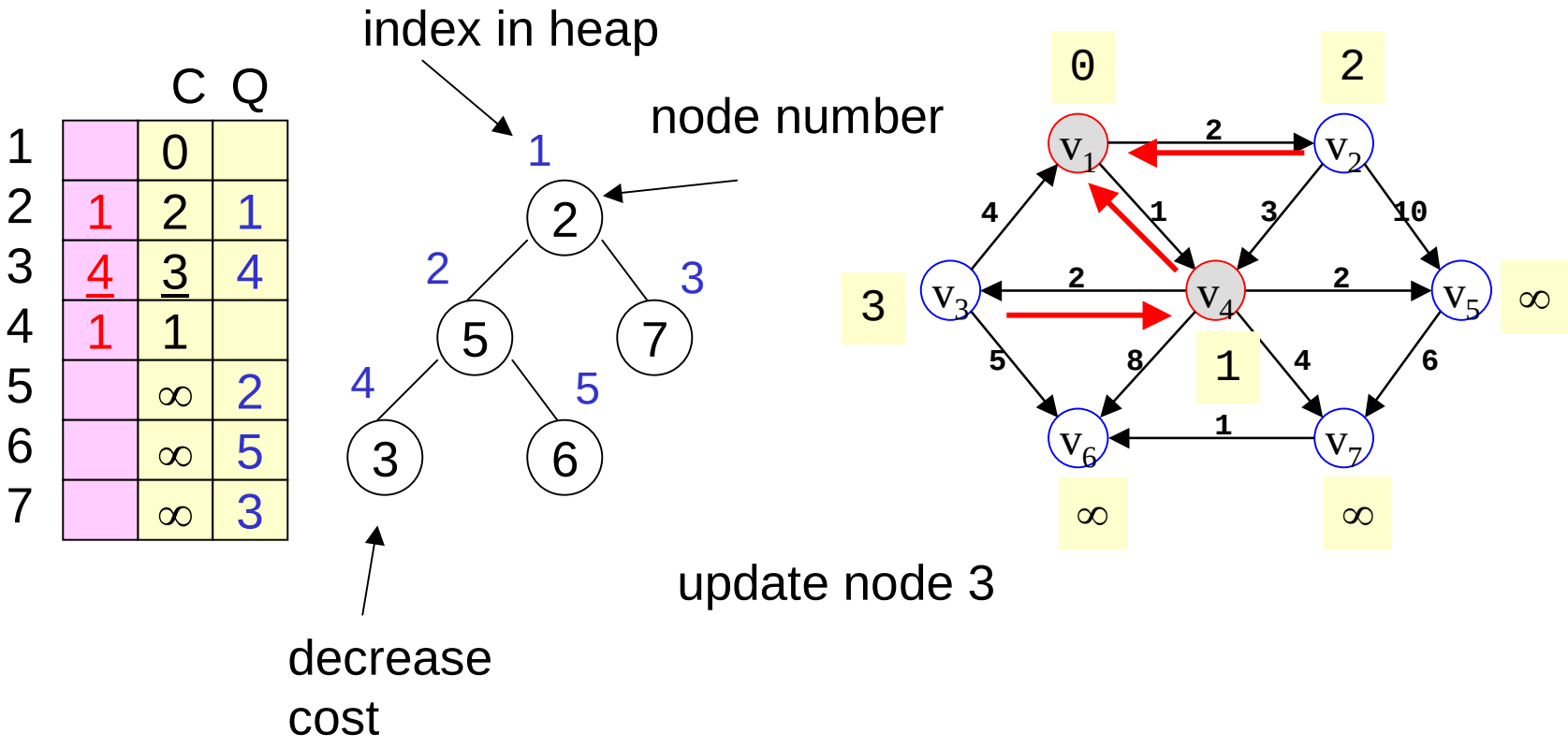
		C	Q
1		0	
2	1	2	1
3		∞	4
4	1	1	
5		∞	2
6		∞	5
7		∞	3





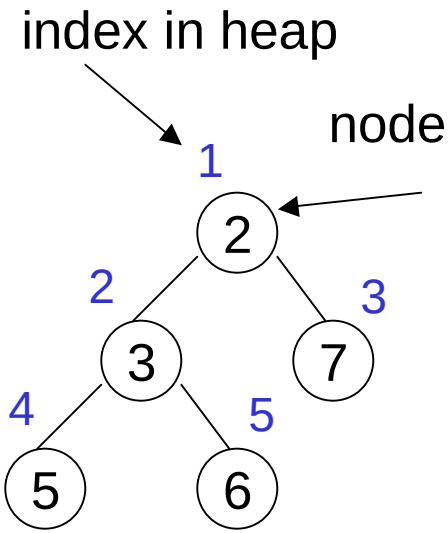
Before the update, but after find min.

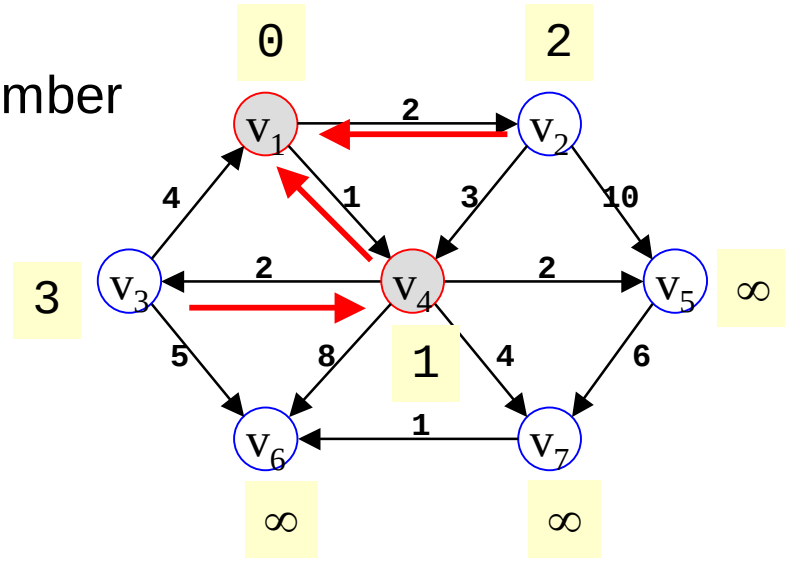
Priority Queue



Priority Queue

		C	Q
1		0	
2	1	2	1
3	4	3	2
4	1	1	
5		∞	4
6		∞	5
7		∞	3





percolate up

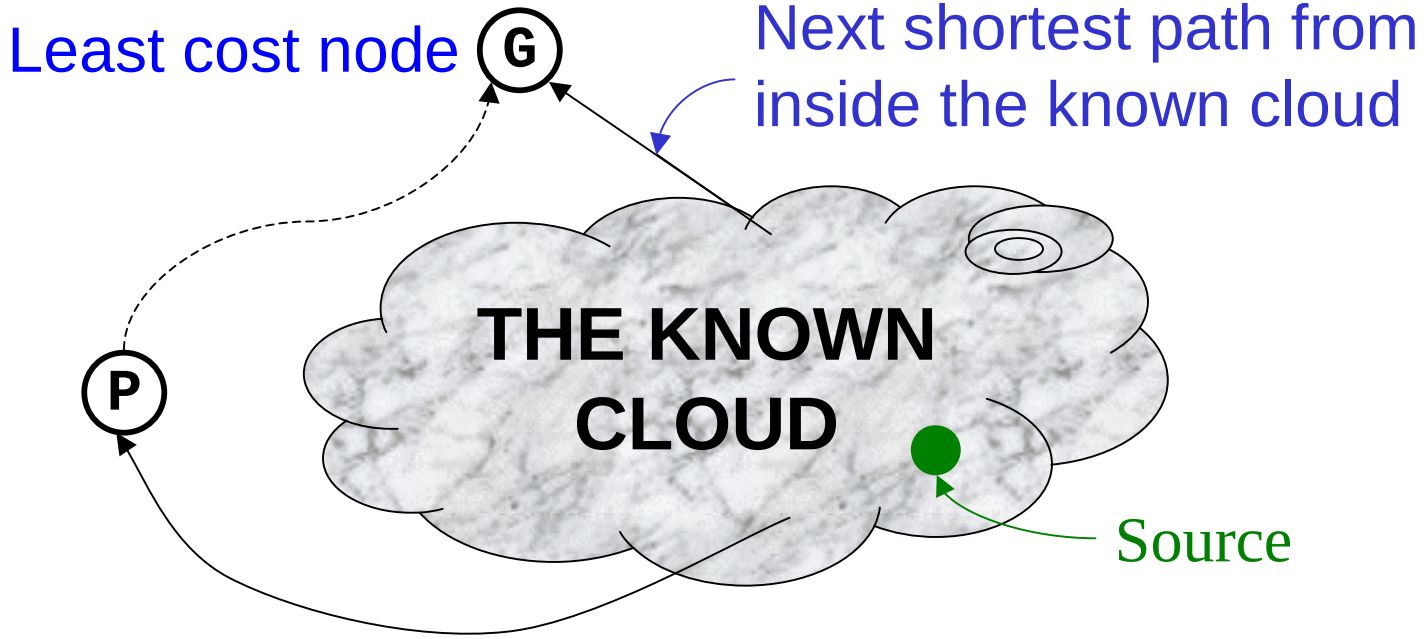
Time Complexity

- n vertices and m edges
- Initialize data structures O(n+m)
- Find min cost vertices O(n log n)
  - n delete mins
- Update costs O(m log n)
  - Potentially m updates
- Update previous pointers O(m)
  - Potentially m updates
- Total time O((n + m) log n) - very fast.

Does It Always Work?

- Dijkstra's algorithm is an example of a greedy algorithm
- Greedy algorithms always make choices that currently seem the best
  - Short-sighted – no consideration of long-term or global issues
  - Locally optimal does not always mean globally optimal
- In Dijkstra's case – choose the least cost node, but what if there is another path through other vertices that is cheaper?

“Cloudy” Proof



- If the path to G is the next shortest path, the path to P must be at least as long. Therefore, any path through P to G cannot be shorter!

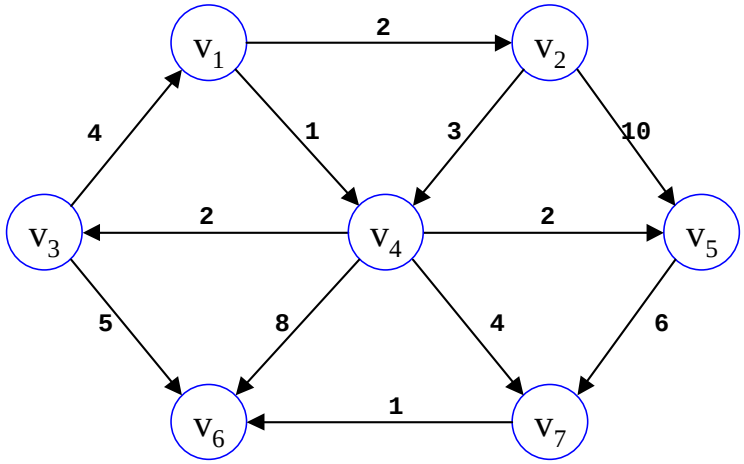
Inside the Cloud (Proof)

- Everything inside the cloud has the correct shortest path
- Proof is by induction on the number of nodes in the cloud:
  - Base case: Initial cloud is just the source with shortest path 0
  - Inductive hypothesis: cloud of k-1 nodes all have shortest paths
  - Inductive step: choose the least cost node G à has to be the shortest path to G (previous slide). Add k-th node G to the cloud

All Pairs Shortest Path

- Given a edge weighted directed graph G = (V,E) find for all u,v in V the length of the shortest path from u to v. Use matrix representation.

C	1	2	3	4	5	6	7
1	0	2	:	1	:	:	:
2	:	0	:	3	10	:	:
3	4	:	0	:	:	5	:
4	:	:	2	0	2	8	4
5	:	:	:	:	0	:	6
6	:	:	:	:	:	0	:
7	:	:	:	:	:	1	0



: = infinity

Matrix Representation

- C[i,j] = the cost of the edge (i,j)
  - C[i,i] = 0 because no cost to stay where you are
  - C[i,j] = infinity (:) if no edge from i to j.

C	1	2	3	4	5	6	7
1	0	2	:	1	:	:	:
2	:	0	:	3	10	:	:
3	4	:	0	:	:	5	:
4	:	:	2	0	2	8	4
5	:	:	:	:	0	:	6
6	:	:	:	:	:	0	:
7	:	:	:	:	:	1	0

Floyd – Warshall Algorithm

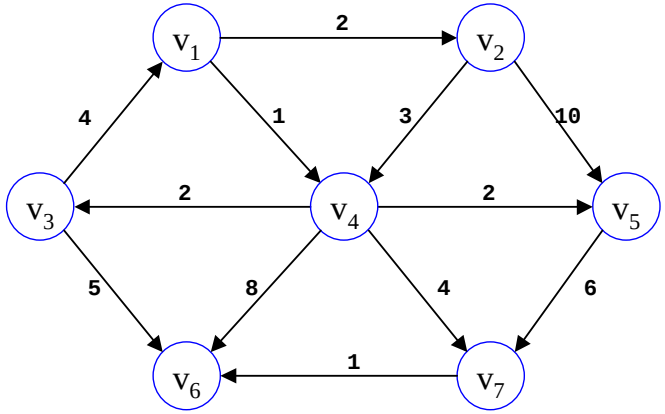
```
All_Pairs_Shortest_Path {
for k = 1 to n do
  for i = 1 to n do
    for j = 1 to n do
      C[i,j] := min(C[i,j], C[i,k] + C[k,j]);
}

Note x + : = : by definition
```

On termination C[i,j] is the length of the shortest path from i to j.

The Computation

C	1	2	3	4	5	6	7
1	0	2	:	1	:	:	:
2	:	0	:	3	10	:	:
3	4	:	0	:	:	5	:
4	:	:	2	0	2	8	4
5	:	:	:	:	0	:	6
6	:	:	:	:	:	0	:
7	:	:	:	:	:	1	0

C	1	2	3	4	5	6	7
1	0	2	3	1	3	6	5
2	9	0	5	3	5	8	7
3	4	6	0	5	4	5	6
4	6	8	2	0	2	5	4
5	:	:	:	:	0	7	6
6	:	:	:	:	:	0	:
7	:	:	:	:	:	1	0



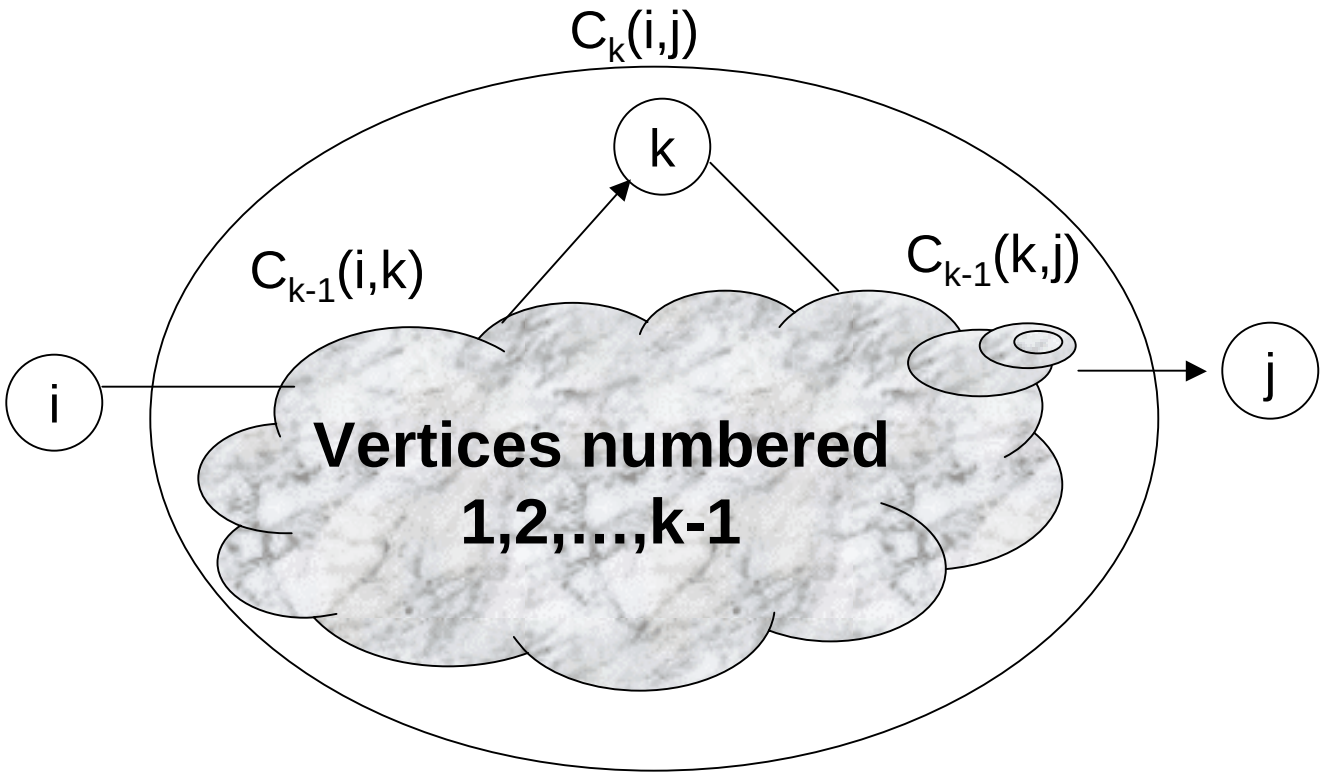
Proof of Correctness

- After the k-th time through the loop C[i,j] is the length of the shortest path that only passes through vertices numbered 1,2,…,k.
  - Let $C_k[i,j]$ be C[i,j] after k time through the loop.
- Base case: k = 0. $C_0[i,j]$ is the cost of an edge that does not pass through any vertices.

Inductive Step

- Assume true for k-1.
  - › A shortest path from i to j that only goes through vertices 1,2, …, k does not go through vertex k at all.
    - $C_k[i,j] = C_{k-1}[i,j]$
  - › All shortest paths from i to j that only goes through vertices 1,2, …, k must go through vertex k.
    - $C_k[i,j] = C_{k-1}[i,k] + C_{k-1}[k,j]$

Cloud Argument



Time Complexity of All Pairs Shortest Path

- n is the number of vertices
- Three nested loops. $O(n^3)$
  - Shortest paths can be found too (see the book).
- Repeated Dijkstra's algorithm
  - $O(n(n+m)\log n)$ (= $O(n^3 \log n)$ for dense graphs).
  - Run Dijkstra starting at each vertex.
  - Dijkstra also gives the shortest paths not just their lengths.