Sorting Summary

CSE 373 - Data Structures

May 15, 2002

Readings and References

- Reading
  - › Sections 7.8-7.11, *Data Structures and Algorithm Analysis in C*, Weiss
- Other References

How fast can we sort?

- Heapsort, Mergesort, and Quicksort all run in O(N log N) best case running time
- Can we do any better?
- Can we believe LaMoC, Inc, which claims to have discovered an O(N log(log N)) general purpose sorting algorithm?
  - The US patent office probably believes it, do you?

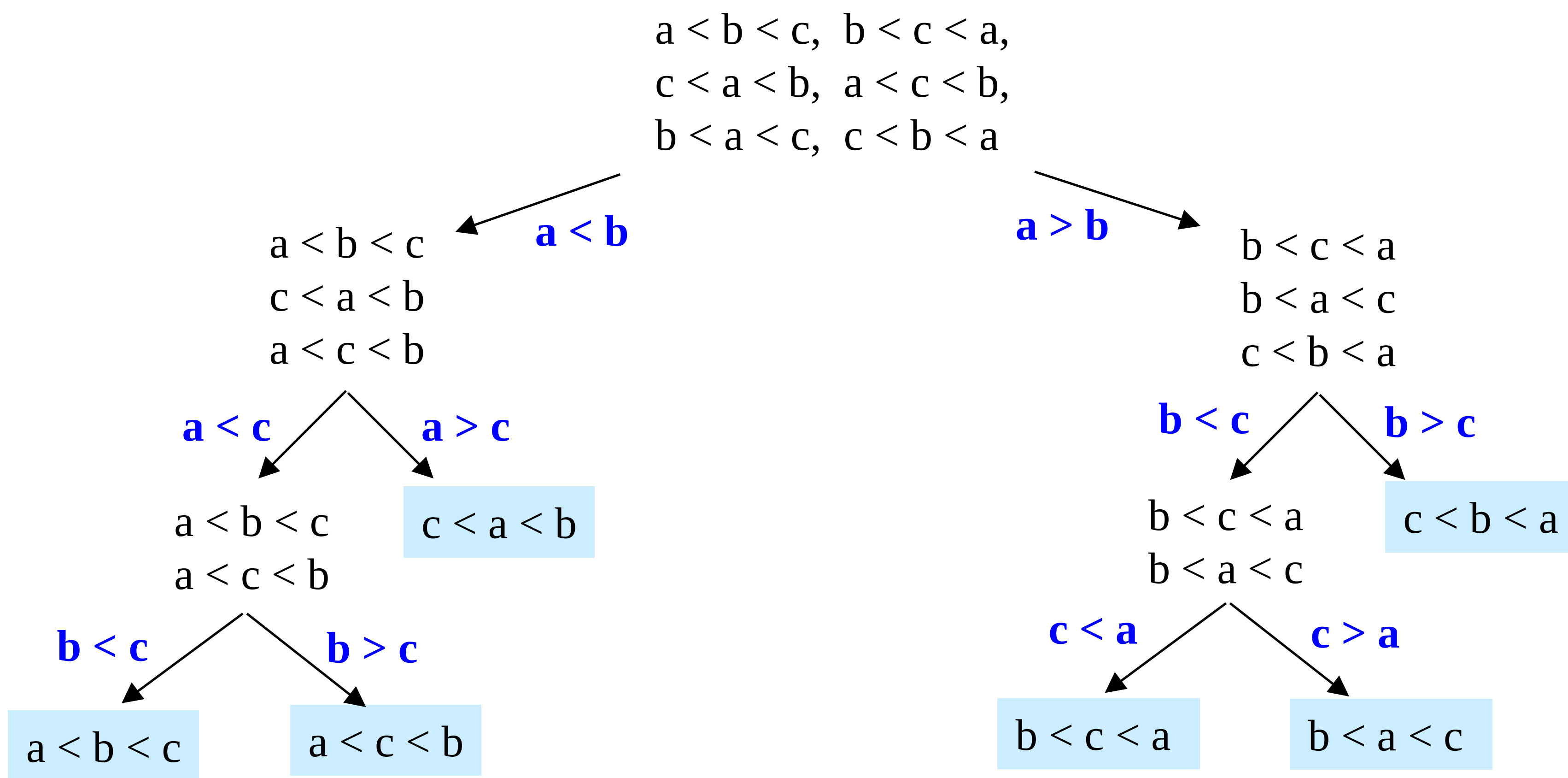
No! (if using comparisons only)

- Recall our basic assumption: we can <u>only compare two elements at a time</u>
  - › we can only reduce the possible solution space by half each time we make a comparison
- Suppose you are given N elements
  - › Assume no duplicates
- How many possible orderings can you get?
  - › Example: a, b, c (N = 3)

Permutations

- How many possible orderings can you get?
  - › Example: a, b, c (N = 3)
  - › (a b c), (a c b), (b a c), (b c a), (c a b), (c b a)
  - › 6 orderings = $3 \cdot 2 \cdot 1 = 3!$ (ie, “3 factorial”)
  - › All the possible permutations of a set of 3 elements
- For N elements
  - › N choices for the first position, (N-1) choices for the second position, …, (2) choices, 1 choice
  - › $N(N-1)(N-2)\cdots(2)(1)=$ <u>N! possible orderings</u>

Decision Tree



The leaves contain all the possible orderings of a, b, c

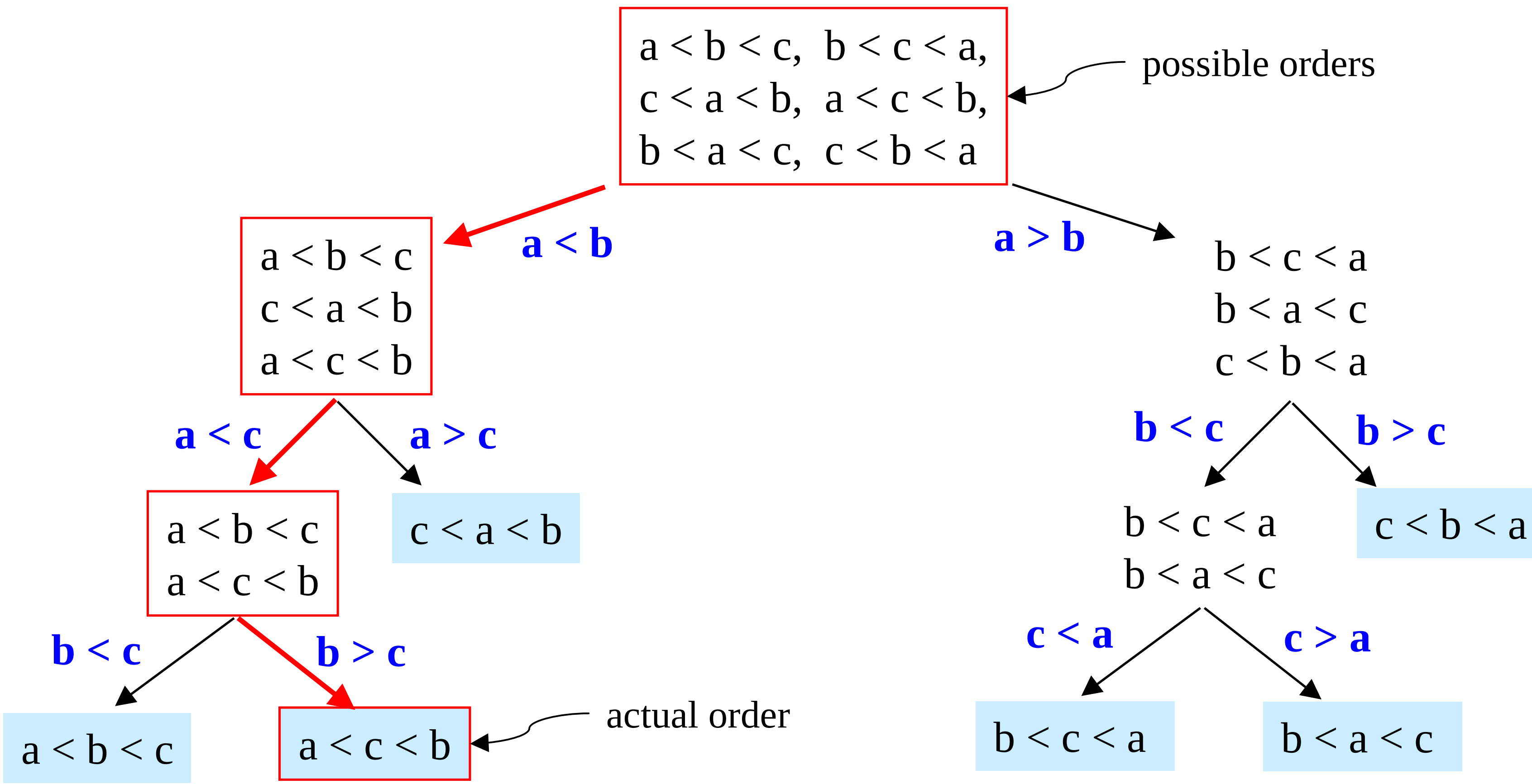
Decision Trees

- A Decision Tree is a Binary Tree such that:
  - › Each node = a set of orderings
    - ie, the remaining solution space
  - › Each edge = 1 comparison
  - › Each leaf = 1 unique ordering
  - › How many leaves for N distinct elements?
    - N!, ie, a leaf for each possible ordering
- Only 1 leaf has the ordering that is the desired correctly sorted arrangement

Decision Trees and Sorting

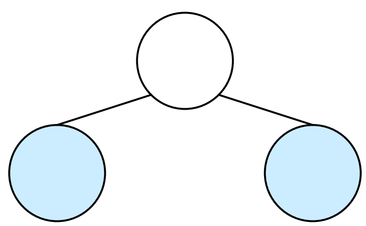
- Every sorting algorithm corresponds to a decision tree
  - › Finds correct leaf by choosing edges to follow
    - ie, by making comparisons
  - › Each decision reduces the possible solution space by one half
- Run time is ≥ maximum no. of comparisons
  - › maximum number of comparisons is the length of the longest path in the decision tree
    - the length of the longest path is the depth of the tree

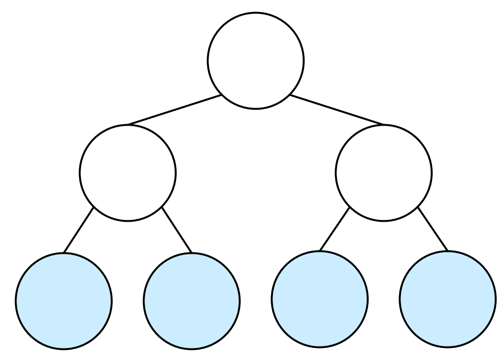
Decision Tree Depth Example



How many leaves on a tree?

- Suppose you have a binary tree of depth d . How many leaves can the tree have?
  - › d = 1 → at most 2 leaves,
  - › d = 2 → at most 4 leaves, etc.





How deep is it, Jim?

- A binary tree of depth d has at most $\mathbf{2^d}$ leaves
  - depth d = 1 → 2 leaves, d = 2 → 4 leaves, etc.
  - Can prove by induction
- The decision tree has L = N! leaves
- Depth d must be deep enough such that $2^d \geq L$
  - and $2^d \geq L$ → $d \geq \log L$
- So the decision tree depth is $d \geq \log(N!)$

$\log(N!)$ is $\Omega(N\log N)$

$$\log(N!) = \log\big(N\cdot(N-1)\cdot(N-2)\cdots(2)\cdot(1)\big)$$

$$= \log N + \log(N-1) + \log(N-2) + \cdots + \log 2 + \log 1$$

select just the first N/2 terms

$$\geq \log N + \log(N-1) + \log(N-2) + \cdots + \log\frac{N}{2}$$

each of the selected terms is ≥ logN/2

$$\geq \frac{N}{2}\log\frac{N}{2}$$

$$\geq \frac{N}{2}(\log N - \log 2) = \frac{N}{2}\log N - \frac{N}{2}$$

$$= \Omega(N\log N)$$

Ω(N log N)

- Run time of any comparison-based sorting algorithm is **Ω(N log N)**
  - › Any sorting algorithm based on comparisons between elements requires **Ω(N log N)** comparisons
- Can never find an O(N log log N) general purpose sorting algorithm
  - › sorry, LaMoC, Inc!
  - › get a clue, patent office

What about bucket sort?

- You may be saying to yourself
  - "But on slide 27 of the List lecture on April 5th, he showed that the bucket sort only takes O(N+B) operations, what's up with that?"
- And I say to you: Advance knowledge of the data lets you do all sorts of magic
  - › perfect hash
  - › bucket sort, radix sort

Bucket Sort: Sorting integers

- Bucket sort: N *integers in the range 0 to B-1*
  - › Array Count has B elements (“buckets”), initialized to 0
  - › Given input integer i, Count[i]++
  - › After reading all **N** numbers go through the **B** buckets and read out the resulting sorted list
  - › **N** operations to read and record the numbers plus **B** operations to recover the sorted numbers

Bucket Sort Run Time?

- What is the running time for sorting N integers?
  - › Running Time: O(B+N)
    - B to zero/scan the array and N to read the input
  - › If B is Θ(N), running time for Bucket sort = **O(N)**
- Doesn’t this violate the O(N log N) lower bound result??
- No – When we do Count[i]++, we are comparing one element with <u>all B elements</u>, not just two elements

Radix Sort: Sorting integers

- Radix sort = multi-pass bucket sort of integers in the range 0 to B^P-1
  - › Bucket-sort from least significant to most significant “digit” (base B)
  - › Use linked list to store numbers that are in same bucket
  - › Requires P*(B+N) operations where P is the number of passes (the number of base B digits in the largest possible input number)
  - › Do P passes instead of using B^P space

Radix Sort Example

data

478
537
9
721
3
38
123
67

Bucket sort by 1's digit

0	1	2	3	4	5	6	7	8	9
	721		3 123				537 67	478 38	9

Bucket sort by 10's digit

0	1	2	3	4	5	6	7	8	9
03 09		721 123	537 38			67	478		

Bucket sort by 100's digit

0	1	2	3	4	5	6	7	8	9
003 009 038 067	123			478	537		721		

This example uses B=10 and base 10 digits for simplicity of demonstration. Larger bucket counts should be used in an actual implementation.

Internal versus External Sorting

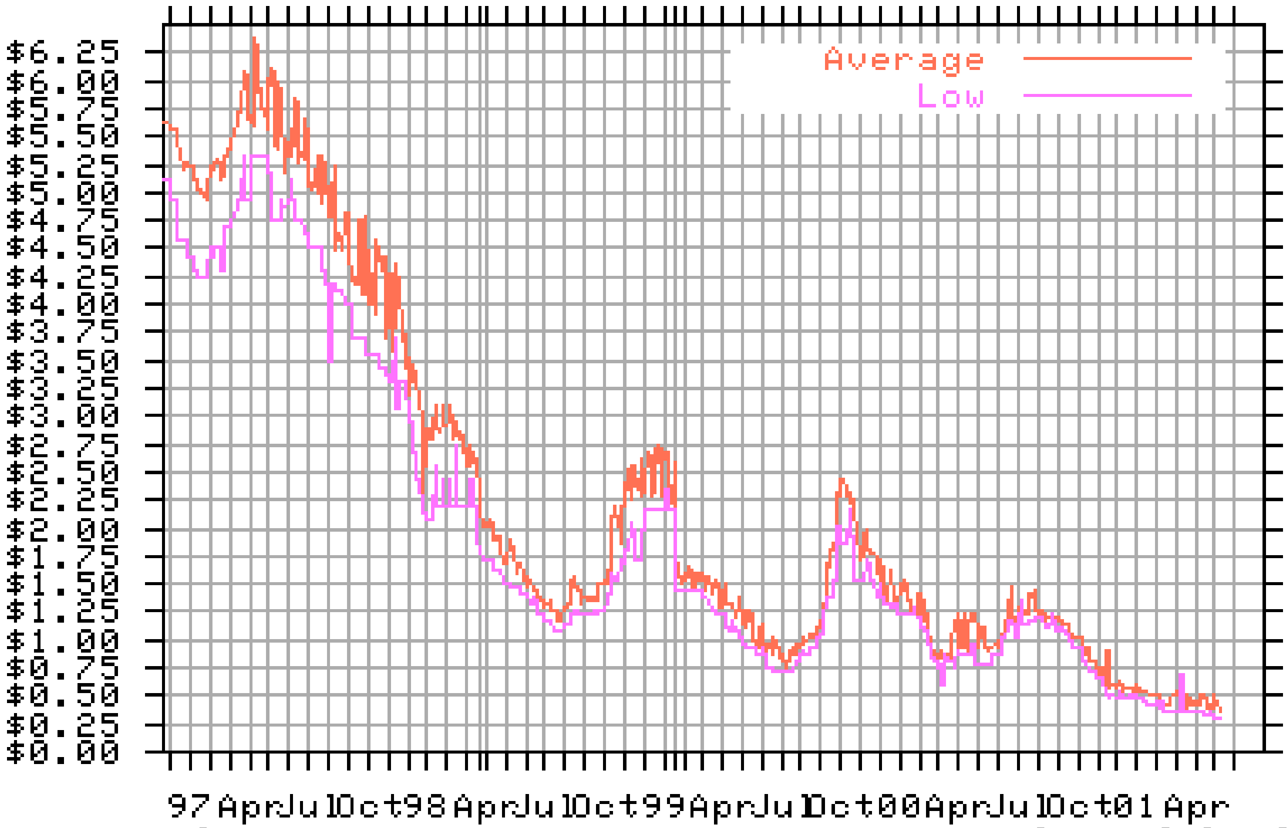
- So far assumed that accessing A[i] is fast – Array A is stored in internal memory (RAM)
  - Algorithms so far are good for internal sorting
- What if A is so large that it doesn't fit in internal memory?
  - Data on disk or tape
  - Delay in accessing A[i] – e.g. need to spin disk and move head

Internal versus External Sorting

- Need sorting algorithms that minimize disk/tape access time
  - › External sorting – Basic Idea:
    - Load chunk of data into RAM, sort, store this "run" on disk/tape
    - Use the Merge routine from Mergesort to merge runs
    - Repeat until you have only one run (one sorted chunk)
    - Text gives some examples
- But … how important is external sorting?

Internal Memory is getting cheap…

Price (in US$) for 1 MB of RAM

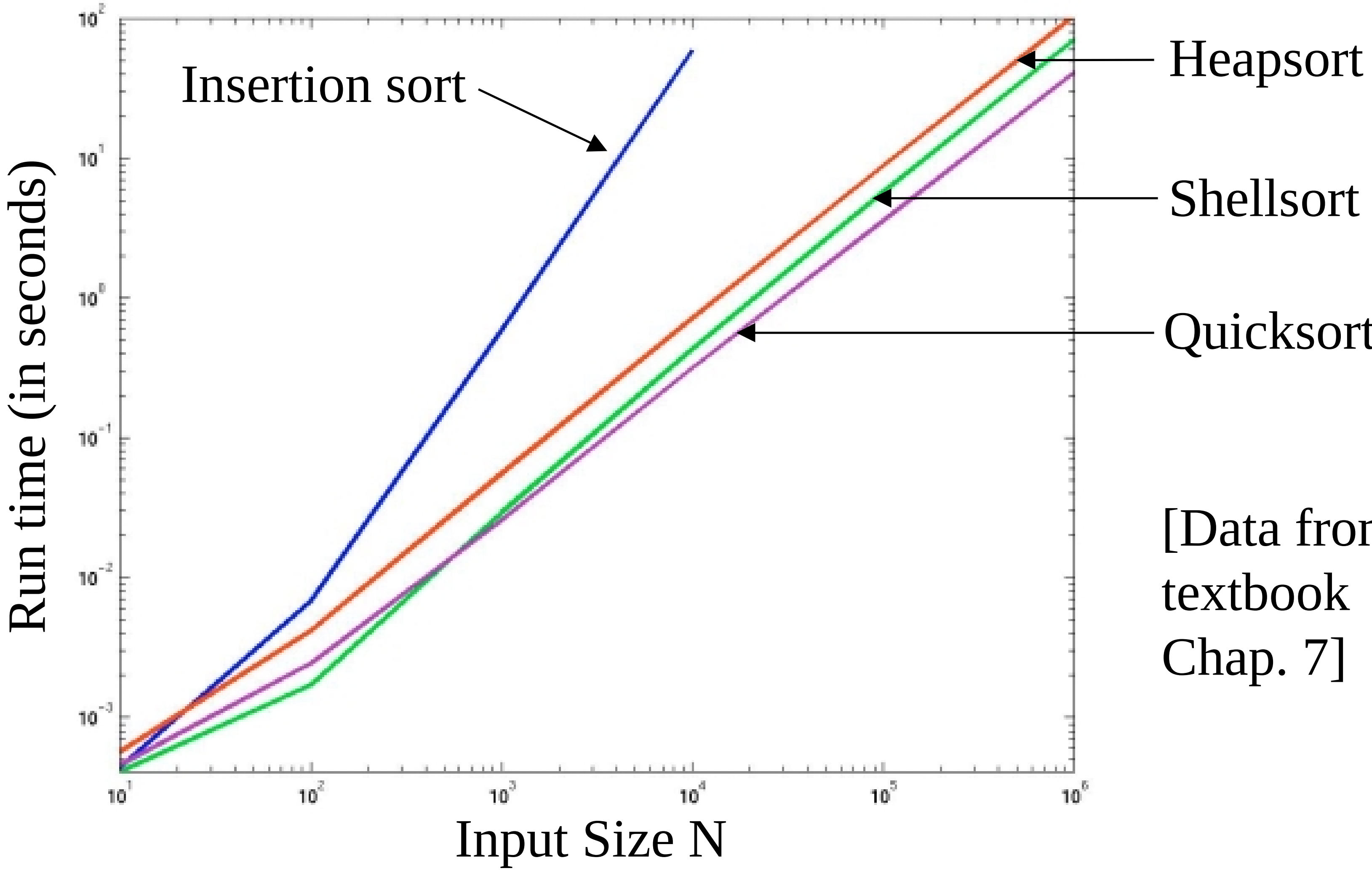


From: http://www.macresource.com/mrp/ramwatch/trend.shtml

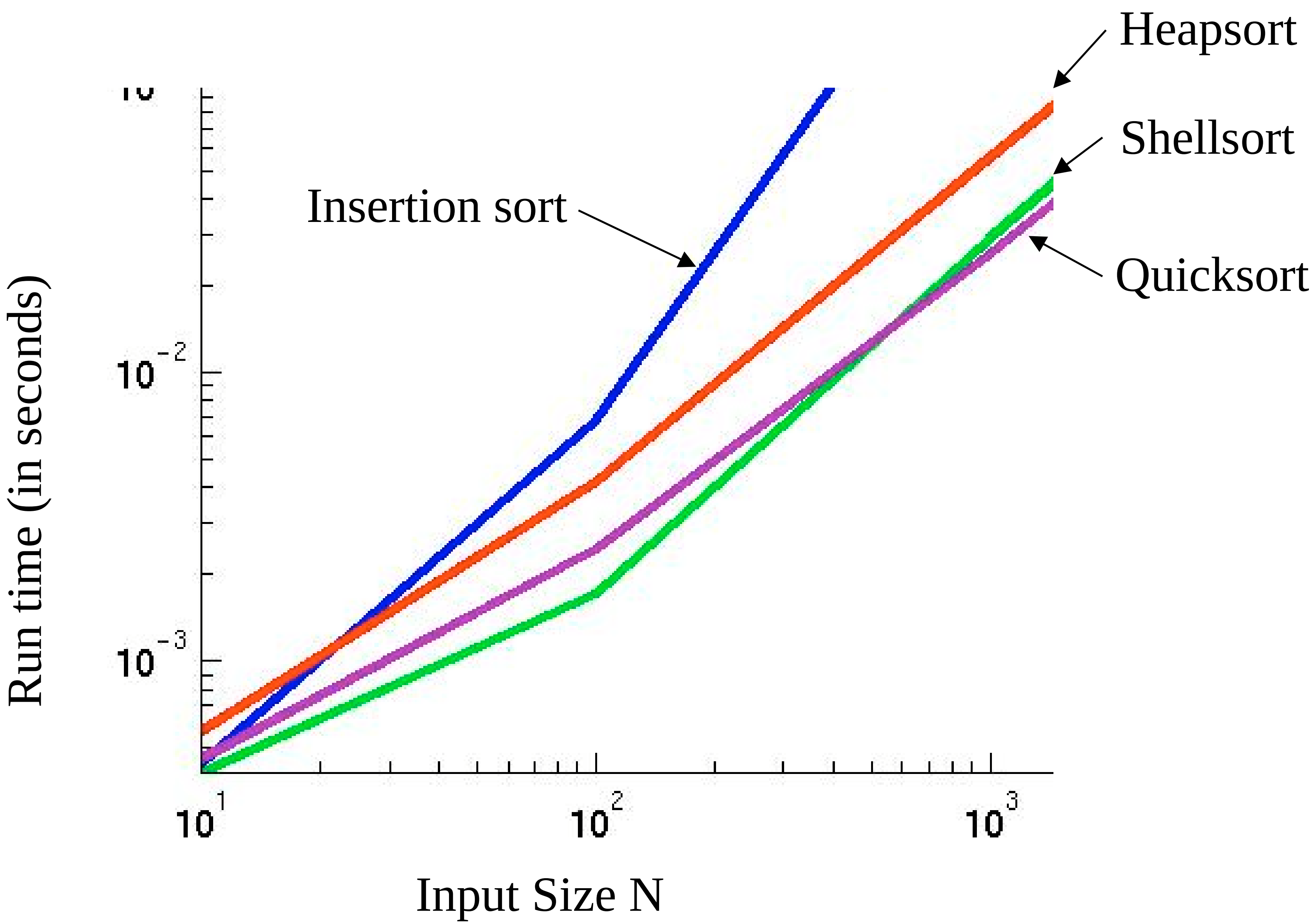
External Sorting

- For most data sets, internal sorting in a large memory space is possible and intricate external sorts are not required
  - › Tapes seldom used these days – random access disks are faster and getting cheaper with greater capacity
  - › Operating systems provide very, very large virtual memory address spaces so it looks like an internal sort, even though you are using the disk
    - be careful though, you can end up doing a lot of disk I/O if you’re not careful

Okay…so let's talk about performance in practice



[Data from textbook Chap. 7]


Heapsort
Shellsort
Insertion sort
Quicksort
Run time (in seconds)
10^{-2}
10^{-3}
10^{1}
10^{2}
10^{3}
Input Size N

Summary of Sorting

- Sorting choices:
  - › $O(N^2)$ – Bubblesort, Selection Sort, Insertion Sort
  - › $O(N^x)$ – Shellsort ($x = 3/2, 4/3, 7/6, 2$, etc. depending on increment sequence)
  - › O(N log N) average case running time:
    - Heapsort: uses 2 comparisons to move data (between children and between child and parent) – may not be fast in practice (see graph)
    - Mergesort: easy to code but uses O(N) extra space
    - Quicksort: fastest in practice but trickier to code, $O(N^2)$ worst case

Practical Sorting

- When N is large, use Quicksort with median3 pivot
- For small N (< 20), the N log N sorts are slower due to extra overhead (larger constants in big-Oh notation)
  - › For N < 20, use Insertion sort
  - › In Quicksort, do insertion sort when sub-array size < 20 (instead of partitioning)
- When you need a sorter
  - › remember the various candidate algorithms
  - › think about the type and quantity of your data
  - › look up an appropriate reference implementation and adapt it to your requirements (ElementType, comparator, etc)