Disjoint Sets

CSE 373 - Data Structures

May 17, 2002

Readings and References

- Reading
  - › Chapter 8, *Data Structures and Algorithm Analysis in C*, Weiss
- Other References

Relations on a set

- Consider the relation “=” between integers
  - › For any integer a, a = a
  - › For integers a and b, a = b means that b = a
  - › For integers a, b, and c, a = b and b = c means that a = c

Relations on a set

- Consider cities connected by two-way roads
  - › Seattle is connected to itself
  - › Seattle is connected to Everett means Everett is connected to Seattle
  - › If Seattle is connected to Everett and Everett is connected to Bellingham, then Seattle is connected to Bellingham
- Consider electrical connections between components on a computer chip

Equivalence Relations

- An equivalence relation R obeys three properties:
  - › reflexive: for any x, xRx is true
  - › symmetric: for any x and y, xRy implies yRx
  - › transitive: for any x, y, and z, xRy and yRz implies xRz
- Preceding relations are all examples of *equivalence relations*

Equivalence Relations

- What are some relations that are not equivalence relations?
  - › What about “<” on integers?
    - not reflexive, not symmetric
  - › What about “≤” on integers?
    - not symmetric
  - › What about “is having an affair with” in a soap opera?
    - Victor IHAAW Ashley IHAAW Brad does not imply Victor IHAAW Brad ∴ not transitive
    - probably not reflexive, although in the soaps, who knows ...

Equivalence Classes & Disjoint Sets

- A specific equivalence relation operator R divides all the elements into disjoint sets of related items
- Let "~" be an equivalence relation
- If a~b, then a and b are in the same equivalence class

Equivalence Class Examples

- If ~ denotes “electrically connected,” then sets of connected components on a computer chip form equivalence classes
- On a map, cites that have two-way roads between them form equivalence classes
  - › as long as you say that reflexive means that just sitting in town satisfies Seattle ~ Seattle
    - path length = 0
  - › We don’t have loop roads that go out and come back
    - path length = 1

Modulo example

- The relation "Modulo N" divides all integers in N equivalence classes.
  - › For example, "*a* **mod** 5" on the integers produces 5 equivalence classes (remainders 0 through 4 when the integers are divided by 5)
    - 0 ~ 5 ~ 10 ~ …
    - 1 ~ 6 ~ 11 ~ …
    - 2 ~ 7 ~ 12 ~ …
    - 3 ~ 8 ~ 13 ~ …
    - 4 ~ 9 ~ 14 ~ …

Problem Definition

- Given a set of elements and some equivalence relation ~ between them, we want to figure out the equivalence classes
- Given an element, we want to find the equivalence class it belongs to
  - E.g. Under mod 5, 13 belongs to the equivalence class of 3
  - E.g. For the map example, want to find the equivalence class of Everett (all the cities it is connected to)

Problem Definition

- Given a new element, want to add it to an equivalence class (union)
  - › Add 18 to the "a mod 5" relation already containing the numbers shown
    - Since 18 ~ 3 ~ 8 ~ 13, perform a union of 18 with equivalence class of 3, 8, and 13
  - › Add Monroe to the city connection relation
    - Everett is connected to Monroe, so add Monroe to the same equivalence class as Everett, Seattle, and Bellingham

Disjoint Set ADT

- Find: Given an element, return the “name” of its equivalence class
    - note that we are finding the equivalence class, not the element
- Union: Given the “names” of two equivalence classes, merge them into one class
    - › may have a new name or one of the two old names

Disjoint Set ADT

- The disjoint set ADT divides elements into equivalence classes and manages the combination of classes depending on the relation of interest
  - › Names of classes are arbitrary e.g. 1 through N, so long as Find returns the same name for 2 elements in the same equivalence class

Disjoint Set ADT Properties

- Disjoint set equivalence property
  - every element belongs to exactly one set (its equivalence class)
- *Dynamic* equivalence property
  - the name of the equivalence class that an element is in may change after a union
  - however, all elements in the class will always have the same equivalence class name

More Formal Definition

- Given a set $U = \{a_1, a_2, \dots , a_n\}$
- Maintain a *partition* of U, a set of subsets (or equivalence classes) of U denoted by $\{S_1, S_2, \dots , S_k\}$ such that:
  - › each pair of subsets S_i and S_j are disjoint: $S_i \cap S_j = \varnothing$
  - › together, the subsets cover U: $U = \bigcup_{i=1}^{k} S_i$
  - › each subset has a unique name
- Union(a, b) creates a new subset which is the union of a's subset and b's subset
- Find(a) returns a unique name for a's subset

Simple array implementation?

- How about an array implementation?
  - › Array A → A[i] holds the class name for element i
  - › E.g. if 18 ~ 3, pick 3 as class name and set A[18] = A[3] = 3
  - › Running time for Find(i)?
    - just return A[i] : O(1)
  - › Running time for Union(i,j)?
    - If first N/2 elements have class name 1 and next N/2 have class name 2, Union(1,2) will need to change class names of N/2 items : O(N)

Linked List Implementation?

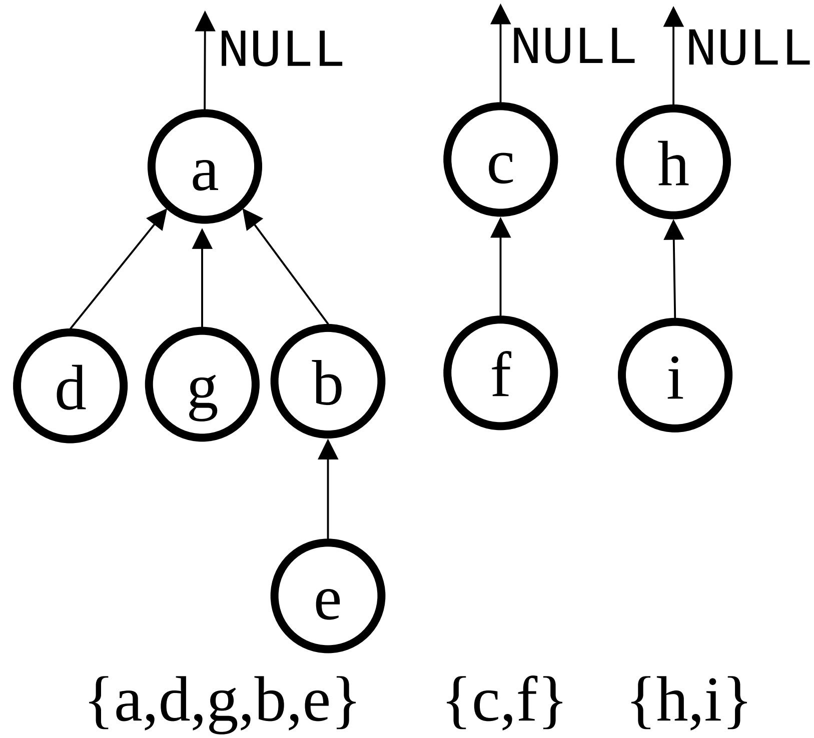
- How about linked lists?
  - › One linked list for each equivalence class
  - › Running time for Find(i)?
    - must scan all lists in worst case : O(N)
  - › Running time for Union(i,j)?
    - just append one list to the other : O(1)
- Tradeoff between Union-Find – cannot do both in O(1) time
  - › M Finds and N-1 Unions (the max)
    - array $O(M + N^2)$ or lists O(MN+N)

Let’s use a new Data Structure

- Intuition: Finding the representative member (= class name) of a set is like the *opposite* of finding a key in a given set
- So, instead of trees with pointers from each node to its children, let’s use trees with a pointer from each node to its parent
- Such trees are known as Up-Trees

Up-Tree Data Structure

- Each equivalence class (or discrete set) is an up-tree with its root as its representative member
- All members of a given set are nodes in that set's up-tree
- Hash table maps input data to the node associated with that data
  - › input string → integer

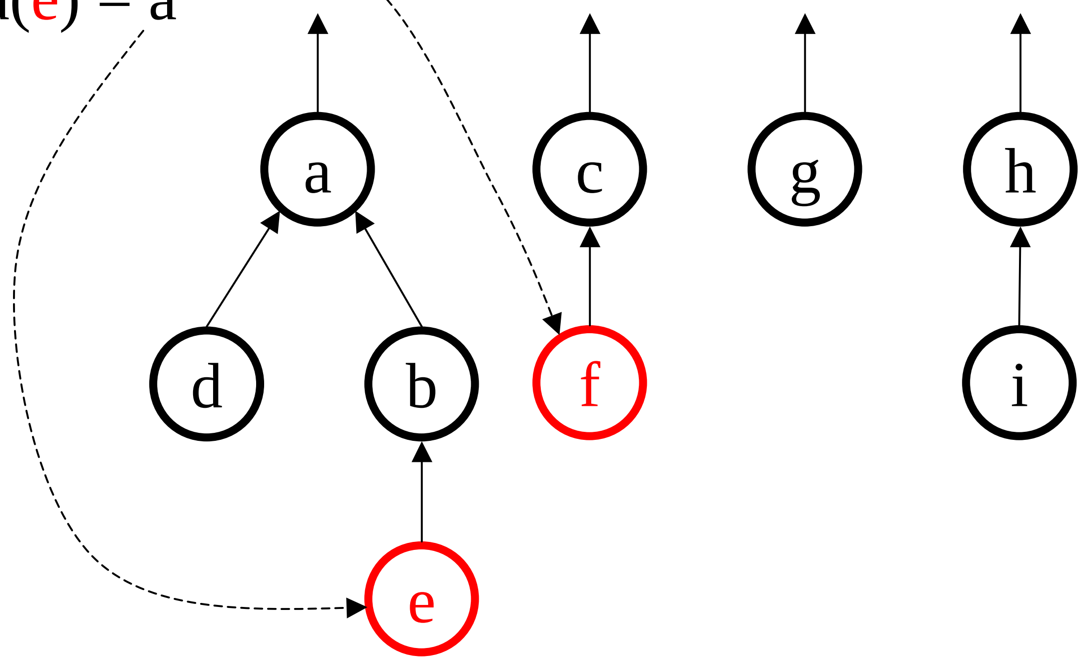


Up-trees are usually **not** binary!

Example of Find

Find: Just traverse from the node to the root.

find(f) = c
find(e) = a

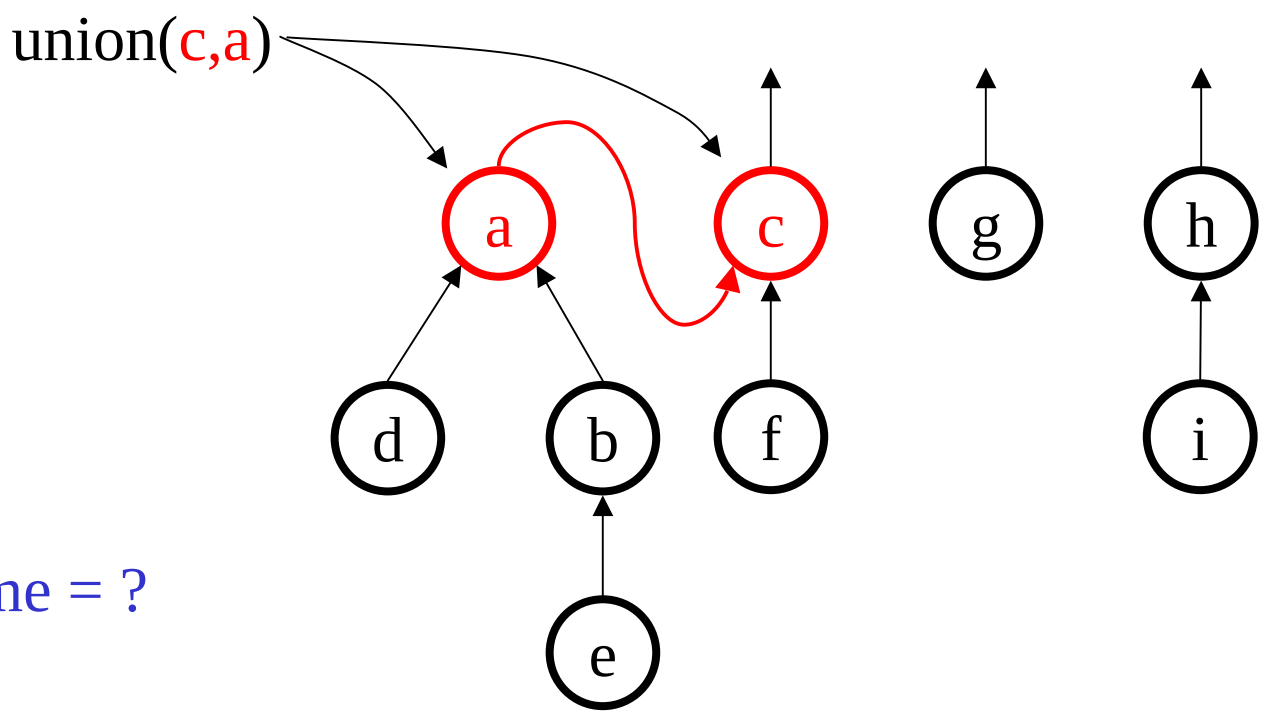


Runtime = ?

Example of Union

Union: Just hang one root from the other.

Now: find(f) = c
find(e) = c



Runtime = ?