Problem 2:

Prove by induction that for all nonnegative integers n
$6(1^2) + 6(2^2) + ... + 6(n^2) = n(n+1)(2n+1)$

Theorem: $6(1^2) + 6(2^2) + ... + 6(n^2) = n(n+1)(2n+1)$ for all $n \geq 0$

Prove: By induction on n.

Basis: $(n = 0)$ $6(0^2) = 0(0 + 1)(0 + 1)$
          Basis is true for case $n = 0$

Induction Hypothesis:
          Assume that the theorem $6(1^2) + 6(2^2) + ... + 6(n^2) = n(n+1)(2n+1)$ for
          integer n that is greater than or equal 0 (basis), or:
          $(n \geq 0)$ $6(1^2) + 6(2^2) + ... + 6(n^2) = n(n+1)(2n+1)$
          Note that the left part is the same as $\sum_{i=0}^{n} 6(i^2)$. We can rewrite the
          relation:
          $(n \geq 0)$ $\sum_{i=0}^{n} 6(i^2) = n(n+1)(2n+1)$

Induction Step:
          To show induction hypothesis is true for all $n \geq 0$, we need to show that it
          is also true for $n + 1$. So, let $k = n + 1$, and show the theorem is true for the
          case k:

$\sum_{i=0}^{k} 6(i^2) = \sum_{i=0}^{n+1} 6(i^2) = \sum_{i=0}^{n} 6(i^2) + 6(n+1)^2$	By Algebra
$= n(n+1)(2n+1) + 6(n+1)^2$	**By I.H.**
$= (n+1)[n(2n+1) + 6(n+1)]$	
$= (n+1)[2n^2 + n + 6n + 6]$	
$= (n+1)(n+2)(2n+3)$	
$= (n+1)((n+1)+1)(2(n+1)+1)$	By Algebra
$= k(k+1)(2k+1)$	By $k = n+1$

          Shown that if the theorem is true for n, it is also true for $n + 1$

Summary: Theorem is valid for $n = 0$ (basis), and by induction step it is therefore
          valid for $n=1$, $n=2$, … Then by principle of mathematical induction
          theorem is valid for all $n \geq 0$.

Note: Valid solution can be much shorter. You really need Induction Hypothesis,
          Basis, Induction Step, and Summary to tie it all together.

Problem 3 (shorter version than problem 2):

Basis: (n=0) $\sum_{i=0}^{0} 4(i^3) = 0 = (0^2)(0+1)^2$
            Basis is true for n = 0

Induction Hypothesis:
            $(n \geq 0)$ Assume $\sum_{i=0}^{n} 4(i^3) = n^2(n+1)^2$

Induction Step:
            Then need to show that the relationship is also true for n + 1:

$$\sum_{i=0}^{n+1} 4(i^3) = \sum_{i=0}^{n} 4(i^3) + 4(n+1)^3$$
$$= n^2(n+1)^2 + 4(n+1)^3 \qquad \text{By I.H.}$$
$$= (n+1)^2[n^2 + 4(n+1)]$$
$$= (n+1)^2[(n+2)^2]$$
$$= (n+1)^2((n+1)+1)^2$$

                        which is the same relationship with
                        n+1 substituted for n.

            Therefore, we've shown that if hypothesis is true for $n \geq 0$, it is also true
            for n+1.

Summary: The induction hypothesis is valid for n = 0 (basis), and by induction step it
            is therefore valid for n = 1, n=2, … Then by principle of mathematical
            induction the induction hypothesis is valid for all $n \geq 0$.

Problem 4:

This problem seemed like a more challenging than previous problems since it involved
proving a function behaves correctly. But it can be done straight forward with induction.

Note: Many steps can be simplified and shortened, as long as you clearly show how you
        applied induction hypothesis to prove the theorem (thus, prove by induction).

First, let us assume that interleaving two equally sized lists A, B should produce a new
list C such that:
          A : [aN, … a2, a1] |A| = N (size of list A is N)
          B : [bN, … b2, b1] |B| = N (size of list B is N)
          C : [aN, bN, …, a2, b2, a1, b1] |C| = N + N = 2N

Then we need to show that the INTERLEAVE algorithm will produce the same results.

Theorem: INTERLEAVE algorithm correctly interleaves the elements of two lists for a pair of equal-length lists of length 0 or more.

Prove: By induction on length N of both lists (since it is given that lists will be of equal length).

Basis: (N = 0) Two input lists have length 0. Algorithm INTERLEAVE returns null on line 1:

```
if (listA == null) return listB;
```

which returns null, or list of length 0. Therefore, the algorithm works correctly for basis case N = 0.

Induction Hypothesis:

(N ≥ 0) Assume INTERLEAVE algorithm correctly interleaves the elements of two lists (listA, listB) of equally-sized lists of length N.

That is:
    listA : [aN,…, a1]
    listB : [bN,…, b1]
    INTERLEAVE(listA, listB) → [aN, bN, … a1, b1]
for any N ≥ 0.

Induction Step:

Then, need to show that INTERLEAVE algorithm will also interleave elements correctly for length N + 1. Let size of listA and listB be N + 1 elements. So,
    listA : [aN+1, …, a1]
    listB : [bN+1, …, b1]

Then, calling INTERLEAVE algorithm with lists listA, listB will produce the following behavior, ordered by evaluation.

1. Since both lists have size N + 1, and N ≥ 0, then will skip:
```
if (listA == null) return listB;
if (listB == null) return listA;
```
(besides, these will evaluate to true only when we reach the base case).

2. Then, INTERLEAVE will be called with lists listA.next, and listB.next. We know that the length of listA and listB is N + 1 elements, then the length of listA.next and listB.next will be (N+1)–1 elements, or N elements. Then this means that INTERLEAVE will be called with both lists of size N. By Induction Hypothesis, we know that the resulting list will be:

    listA.next : [aN, …, a1]

listB.next : [bN, …, b1]
INTERLEAVE(listA.next, listB.next) → [aN, bN, … a1, b1]

3. Once recursive call to INTERLEAVE is returned, we add listB.data to the front of the list. Then the resulting list will be:
[bN+1, aN, bN, …, a1, b1]

4. Afterwards, we add listA.data to the front of the list. Then the resulting list will be:

[aN+1, bN+1, aN, bN, …, a1, b1]
which is the expected result of INTERLEAVE of N+1 elements that we are trying to show.

5. The INTERLEAVE algorithm then terminates, and correctly interleaved list is returned.

Therefore, we've shown that if the theorem is true for lists length N, it is also true for lists length N + 1.

Summary: Theorem is valid for N = 0 (basis), and by induction step it is therefore valid for N=1, N=2, … Then by principle of mathematical induction theorem is valid for all N ≥ 0, meaning that INTERLEAVE algorithm correctly interleaves the elements of two lists for a pair of equal-length lists of length 0 or more.

Practice Midterm Induction Problem:

```
Prove by induction that the number of edges in a tree with
n nodes is exactly n - 1, for all n > 0. Clearly mark your
basis, induction hypothesis, and induction step.

This is a very simple problem once you see the solution, so
here it goes. Commonly asked question was, what is the
number of edges? Think of it as the number of connecting
lines between nodes that form a tree. Also remember that a
node can be connected to a tree with only one edge!
```

For example: 3 nodes; A, B, C. Number of edges E = 2 (BA
and AC)

```
   A
  / \
 B   C
```
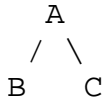
Proof that number of edges E in a tree is N – 1, or E = N –
1.

Prove: By induction on number of nodes N.

Basis: (N = 1) A single root node. There are no edges, or
    E= 1 – 1 = 0
    Basis is true for N = 1

Induction Hypothesis:
    (N ≥ 1) Assume number of edges in a tree is $E = N - 1$ for all N≥1, where N is
    number of nodes.

Induction Step:
    Need to show that if hypothesis is true for N, then it should also be true for N + 1.
    Here is how we can do that:

    Lets take a tree T_0 composed of N nodes. By induction hypothesis we know that
    the number of edges is $E_0 = N - 1$.

    Now lets add another node to T_0 tree, so that we now have N + 1 nodes. Since in a
    valid tree a node can be connected to a tree with only one edge, we know that the
    number of edges in the formed tree is one greater than it was before with N nodes.
    Or,
    $E = E_0 + 1$ for N + 1 nodes.

    Notice that that can be rewritten as

        $E = (N - 1) + 1 = (N + 1) - 1$ for N + 1 nodes,

    which is the relationship we were trying to show. Therefore, we've shown that if
    hypothesis is true for N ≥ 1, it is also true for N+1.

Summary:
    Hypothesis is valid for N = 1 (basis), and by induction step it is therefore valid for
    N=2, N=3, … Then by principle of mathematical induction theorem is valid for
    all N ≥ 1, or Number of edges in a tree of N nodes is $E = N - 1$.