AVL Trees

CSE 373
Data Structures

Readings

- Reading
  - › Goodrich and Tamassia, Chapter 9

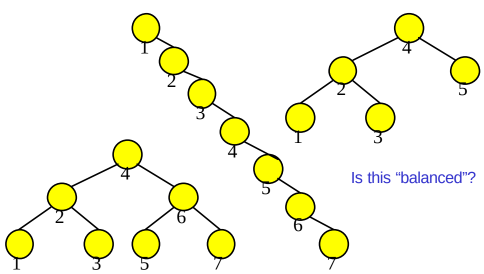
Binary Search Tree - Best Time

- All BST operations are O(d), where d is tree depth
- minimum d is $d = \lfloor \log_2 N \rfloor$ for a binary tree with N nodes
  - › What is the best case tree?
  - › What is the worst case tree?
- So, best case running time of BST operations is O(log N)

Binary Search Tree - Worst Time

- Worst case running time is O(N)
  - › What happens when you Insert elements in ascending order?
    - Insert: 2, 4, 6, 8, 10, 12 into an empty BST
  - › Problem: Lack of "balance":
    - compare depths of left and right subtree
  - › Unbalanced degenerate tree

Balanced and unbalanced BST



Approaches to balancing trees

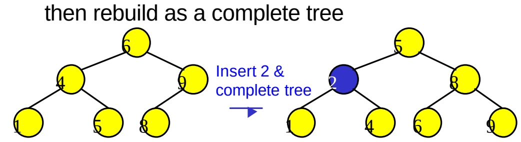
- Don't balance
  - › May end up with some nodes very deep
- Strict balance
  - › The tree must always be balanced perfectly
- Pretty good balance
  - › Only allow a little out of balance
- Adjust on access
  - › Self-adjusting

Balancing Binary Search Trees

- Many algorithms exist for keeping binary search trees balanced
  - › Adelson-Velskii and Landis (AVL) trees (height-balanced trees)
  - › Splay trees and other self-adjusting trees
  - › B-trees and other multiway search trees

Perfect Balance

- Want a complete tree after every operation
  - › tree is full except possibly in the lower right
- This is expensive
  - › For example, insert 2 in the tree on the left and then rebuild as a complete tree

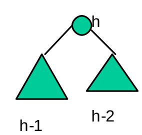


AVL - Good but not Perfect Balance

- AVL trees are height-balanced binary search trees
- Balance factor of a node
  - › height(left subtree) - height(right subtree)
- An AVL tree has balance factor calculated at every node
  - › For every node, heights of left and right subtree can differ by no more than 1
  - › Store current heights in each node

Height of an AVL Tree

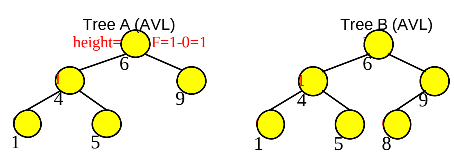
- N(h) = minimum number of nodes in an AVL tree of height h.
- Basis
  - › N(0) = 1, N(1) = 2
- Induction
  - › N(h) = N(h-1) + N(h-2) + 1
- Solution (recall Fibonacci analysis)
  - › $N(h) \geq \phi^h$ ($\phi \approx 1.62$)



Height of an AVL Tree

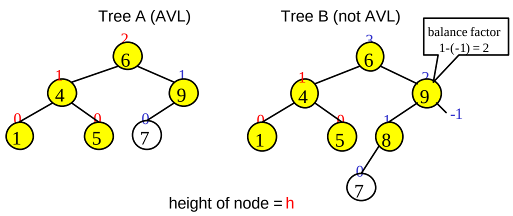
- $N(h) \geq \phi^h$ ($\phi \approx 1.62$)
- Suppose we have n nodes in an AVL tree of height h.
  - › $n \geq N(h)$ (because N(h) was the minimum)
  - › $n \geq \phi^h$ hence $\log_\phi n \geq h$ (relatively well balanced tree!!)
  - › $h \leq 1.44 \log_2 n$ (i.e., Find takes O(logn))

Node Heights



height of node = h
balance factor = h_{left}-h_{right}
empty height = -1

Node Heights after Insert 7

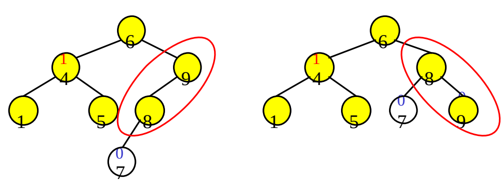


height of node = h
balance factor = h_{left}-h_{right}
empty height = -1

Insert and Rotation in AVL Trees

- Insert operation may cause balance factor to become 2 or –2 for some node
  - › only nodes on the path from insertion point to root node have possibly changed in height
  - › So after the Insert, go back up to the root node by node, updating heights
  - › If a new balance factor (the difference h_{left}-h_{right}) is 2 or –2, adjust tree by *rotation* around the node

Single Rotation in an AVL Tree



Insertions in AVL Trees

Let the node that needs rebalancing be α.

There are 4 cases:

Outside Cases (require single rotation) :

1. Insertion into left subtree of left child of α.
2. Insertion into right subtree of right child of α.

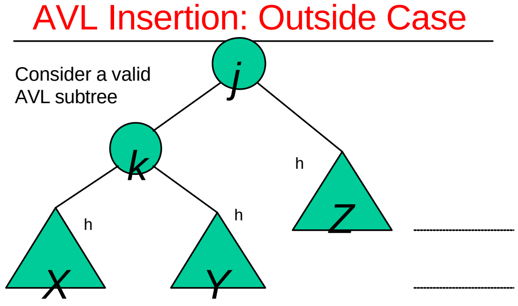
Inside Cases (require double rotation) :

3. Insertion into right subtree of left child of α.
4. Insertion into left subtree of right child of α.

The rebalancing is performed through four separate rotation algorithms.

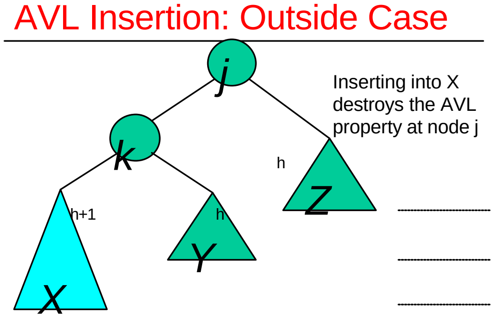
AVL Insertion: Outside Case

Consider a valid AVL subtree



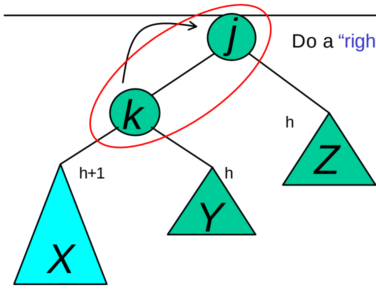
AVL Insertion: Outside Case

Inserting into X destroys the AVL property at node j



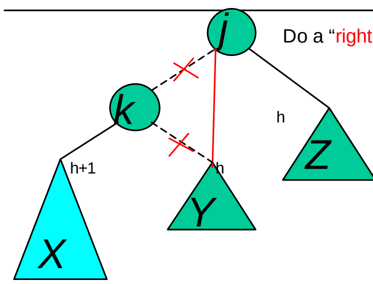
AVL Insertion: Outside Case

Do a "right rotation"



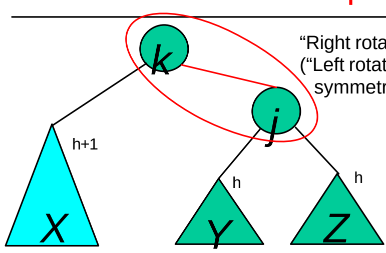
Single right rotation

Do a "right rotation"



Outside Case Completed

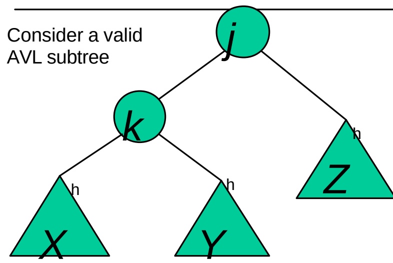
"Right rotation" done!
("Left rotation" is mirror symmetric)



AVL property has been restored!

AVL Insertion: Inside Case

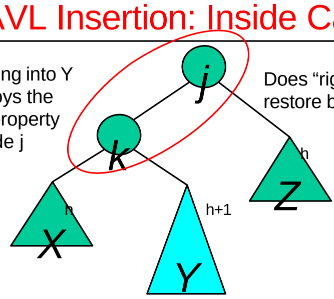
Consider a valid AVL subtree



AVL Insertion: Inside Case

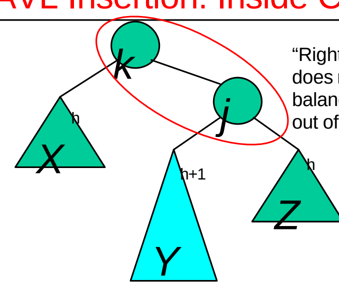
Inserting into Y destroys the AVL property at node j

Does "right rotation" restore balance?



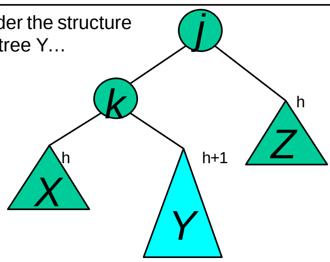
AVL Insertion: Inside Case

"Right rotation" does not restore balance… now k is out of balance



AVL Insertion: Inside Case

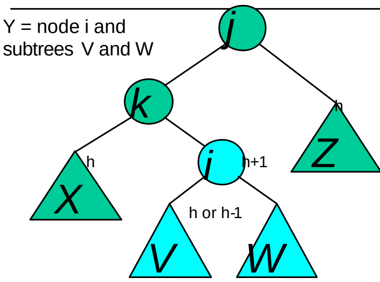
Consider the structure of subtree Y…



4/7/2004 CSE 373 - SP 04 – AVL Trees 25

AVL Insertion: Inside Case

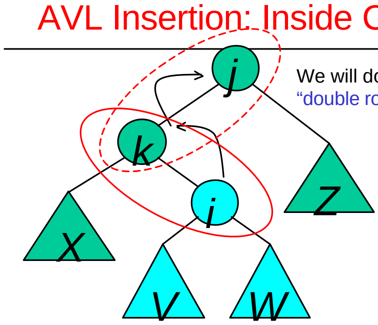
Y = node i and subtrees V and W



4/7/2004 CSE 373 - SP 04 – AVL Trees 26

AVL Insertion: Inside Case

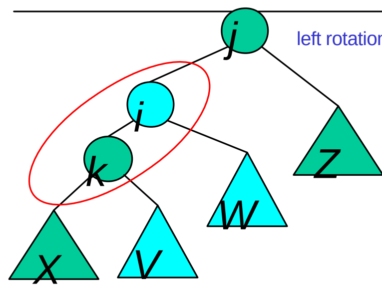
We will do a left-right "double rotation" . . .



4/7/2004 CSE 373 - SP 04 – AVL Trees 27

Double rotation : first rotation

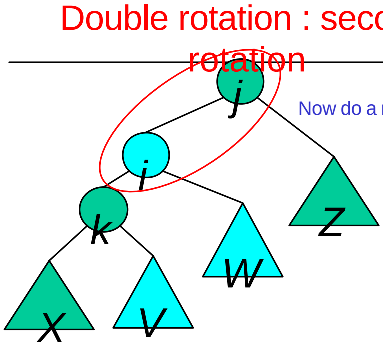
left rotation complete



4/7/2004 CSE 373 - SP 04 – AVL Trees 28

Double rotation : second rotation

Now do a right rotation

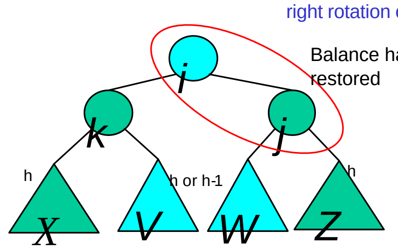


4/7/2004 CSE 373 - SP 04 – AVL Trees 29

Double rotation : second rotation

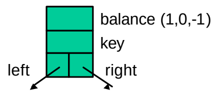
right rotation complete

Balance has been restored



4/7/2004 CSE 373 - SP 04 – AVL Trees 30

Implementation



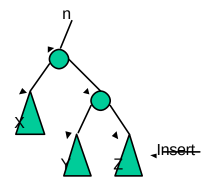
No need to keep the height; just the difference in height, i.e. the balance factor; this has to be modified on the path of insertion even if you don't perform rotations

Once you have performed a rotation (single or double) you won't need to go back up the tree

Single Rotation

```
RotateFromRight(n : reference node pointer) {
p : node pointer;
p := n.right;
n.right := p.left;
p.left := n;
n := p
}
```

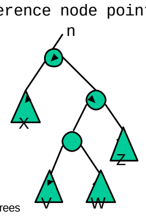
You also need to modify the heights or balance factors of n and p



Double Rotation

- Implement Double Rotation in two lines.

```
DoubleRotateFromRight(n : reference node pointer) {
????
}
```



Insertion in AVL Trees

- Insert at the leaf (as for all BST)
  - only nodes on the path from insertion point to root node have possibly changed in height
  - So after the Insert, go back up to the root node by node, updating heights
  - If a new balance factor (the difference h_{left}-h_{right}) is 2 or –2, adjust tree by *rotation* around the node

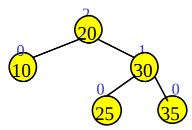
Insert in BST

```
Insert(T : reference tree pointer, x : element) : integer {
if T = null then
  T := new tree; T.data := x; return 1;//the links to
                                        //children are null
case
  T.data = x : return 0; //Duplicate do nothing
  T.data > x : return Insert(T.left, x);
  T.data < x : return Insert(T.right, x);
endcase
}
```


Insert in AVL trees

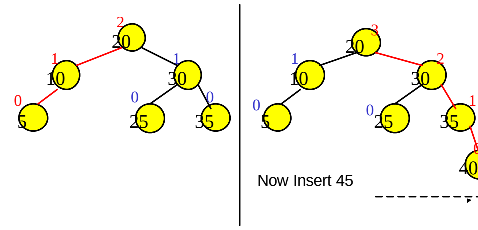
```
Insert(T : reference tree pointer, x : element) : {
if T = null then
  {T := new tree; T.data := x; height := 0; return;}
case
  T.data = x : return ; //Duplicate do nothing
  T.data > x : Insert(T.left, x);
             if ((height(T.left)- height(T.right)) = 2){
                if (T.left.data > x ) then //outside case
                       T = RotatefromLeft (T);
                else                    //inside case
                       T = DoubleRotatefromLeft (T);}
  T.data < x :  Insert(T.right, x);
              code similar to the left case
Endcase
  T.height := max(height(T.left),height(T.right)) +1;
  return;
}
```

Example of Insertions in an AVL Tree

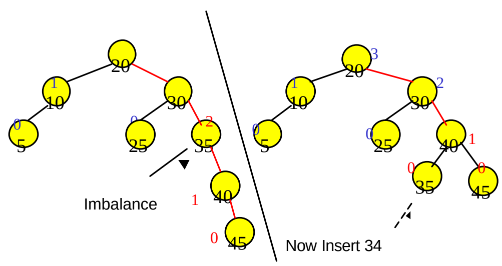


Insert 5, 40

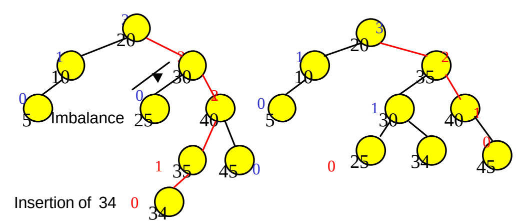
Example of Insertions in an AVL Tree



Single rotation (outside case)



Double rotation (inside case)



AVL Tree Deletion

- Similar but more complex than insertion
  - › Rotations and double rotations needed to rebalance
  - › Imbalance may propagate upward so that many rotations may be needed.

Pros and Cons of AVL Trees

Arguments for AVL trees:

1. Search is O(log N) since AVL trees are always balanced.
2. Insertion and deletions are also O(logn)
3. The height balancing adds no more than a constant factor to the speed of insertion.

Arguments against using AVL trees:

1. Difficult to program & debug; more space for balance factor.
2. Asymptotically faster but rebalancing costs time.
3. Most large searches are done in database systems on disk and use other structures (e.g. B-trees).
4. May be OK to have O(N) for a single operation if total run time for many consecutive operations is fast (e.g. Splay trees).

Double Rotation Solution

```
DoubleRotateFromRight(n : reference node pointer) {
RotateFromLeft(n.right);
RotateFromRight(n);
}
```