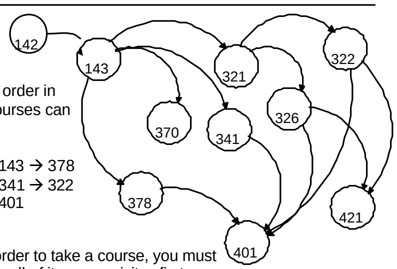
Directed Graph Algorithms

CSE 373

Readings

- Reading
  - › Goodrich and Tamassia, chapter 12

Topological Sort



Problem: Find an order in which all these courses can be taken.

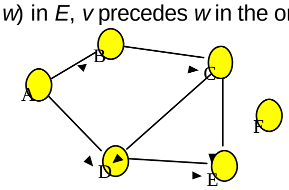
Example: 142 → 143 → 378 → 370 → 321 → 341 → 322 → 326 → 421 → 401

In order to take a course, you must take all of its prerequisites first

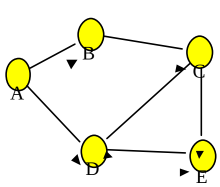
Topological Sort

Given a digraph $G = (V, E)$, find a linear ordering of its vertices such that:

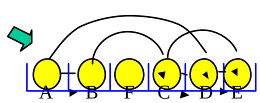
for any edge (v, w) in E, v precedes w in the ordering



Topo sort - good example

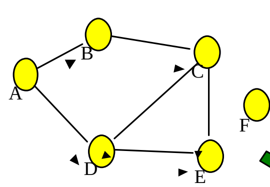


Any linear ordering in which all the arrows go to the right is a valid solution

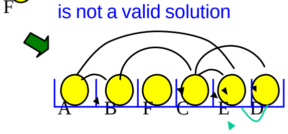


Note that F can go anywhere in this list because it is not connected. Also the solution is not unique.

Topo sort - bad example



Any linear ordering in which an arrow goes to the left is not a valid solution



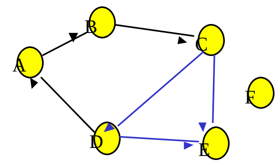
NO!

Paths and Cycles

- Given a digraph G = (V,E), a path is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that:
  - (v_i, v_{i+1}) in E for $1 \le i < k$
  - path length = number of edges in the path
  - path cost = sum of costs of each edge
- A path is a cycle if :
  - $k > 1$; $v_1 = v_k$
- G is acyclic if it has no cycles.

Only acyclic graphs can be topo. sorted

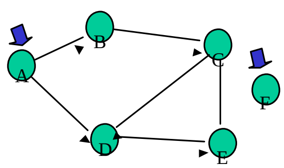
- A directed graph with a cycle cannot be topologically sorted.



Topo sort algorithm - 1

Step 1: Identify vertices that have no incoming edges

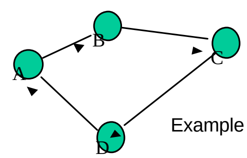
- The "in-degree" of these vertices is zero



Topo sort algorithm - 1a

Step 1: Identify vertices that have no incoming edges

- If *no such vertices*, graph has only cycle(s) (cyclic graph)
- Topological sort not possible – Halt.

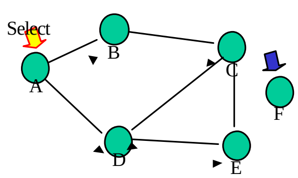


Example of a cyclic graph

Topo sort algorithm - 1b

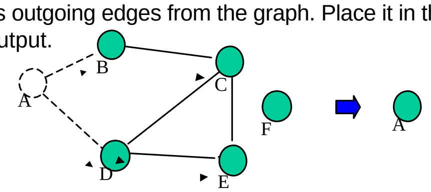
Step 1: Identify vertices that have no incoming edges

- Select one such vertex



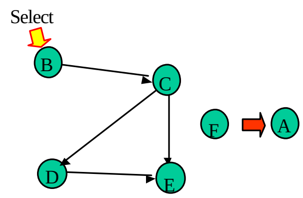
Topo sort algorithm - 2

Step 2: Delete this vertex of in-degree 0 and all its outgoing edges from the graph. Place it in the output.



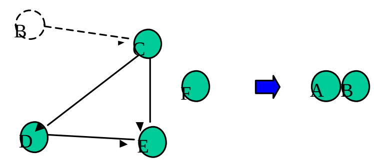
Continue until done

Repeat Step 1 and Step 2 until graph is empty



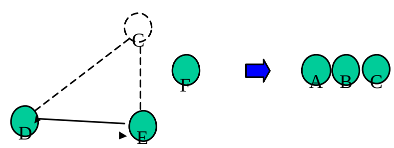
B

Select B. Copy to sorted list. Delete B and its edges.



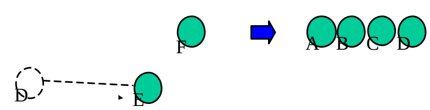
C

Select C. Copy to sorted list. Delete C and its edges.



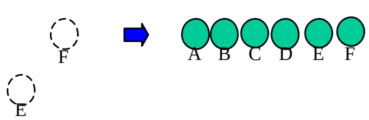
D

Select D. Copy to sorted list. Delete D and its edges.

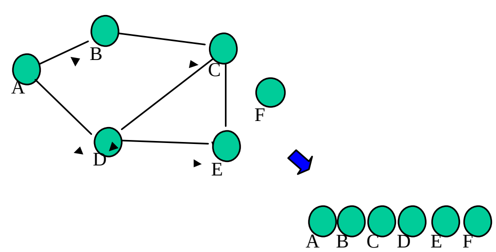


E, F

Select E. Copy to sorted list. Delete E and its edges.
Select F. Copy to sorted list. Delete F and its edges.

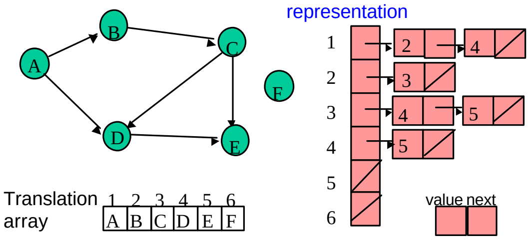


Done



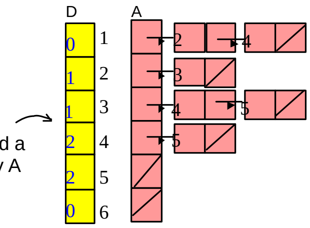
Implementation

Assume adjacency list representation



Calculate In-degrees

In-Degree array; or add a field to array A

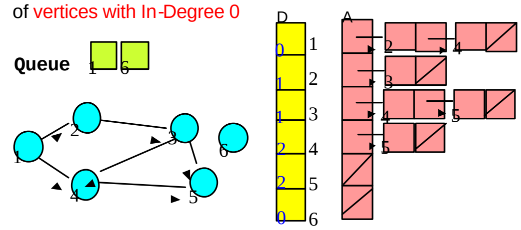


Calculate In-degrees

```
for i = 1 to n do D[i] := 0; endfor
for i = 1 to n do
  x := A[i];
  while x ≠ null do
    D[x.value] := D[x.value] + 1;
    x := x.next;
  endwhile
endfor
```

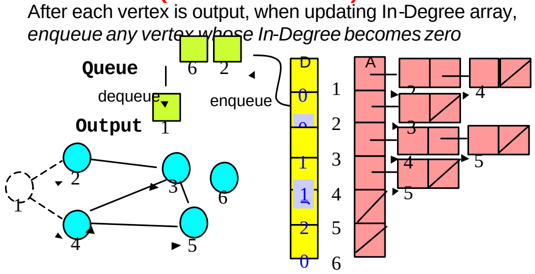

Maintaining Degree 0 Vertices

Key idea: Initialize and maintain a *queue (or stack)* of vertices with In-Degree 0



Topo Sort using a Queue (breadth-first)

After each vertex is output, when updating In-Degree array, *enqueue any vertex whose In-Degree becomes zero*



Topological Sort Algorithm

1. Store each vertex's In-Degree in an array D
2. Initialize queue with all "in-degree=0" vertices
3. While there are vertices remaining in the queue:
   (a) Dequeue and output a vertex
   (b) Reduce In-Degree of all vertices adjacent to it by 1
   (c) Enqueue any of these vertices whose In-Degree became zero
4. If all vertices are output then success, otherwise there is a cycle.

Some Detail

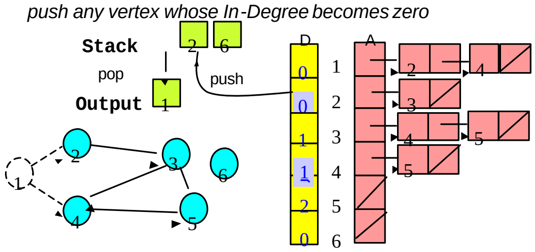
```
Main Loop
while notEmpty(Q) do
  x := Dequeue(Q)
  Output(x)
  y := A[x];
  while y ≠ null do
    D[y.value] := D[y.value] – 1;
    if D[y.value] = 0 then Enqueue(Q,y.value);
    y := y.next;
  endwhile
endwhile
```


Topological Sort Analysis

- Initialize In-Degree array: O(|V| + |E|)
- Initialize Queue with In-Degree 0 vertices: O(|V|)
- Dequeue and output vertex:
  - › |V| vertices, each takes only O(1) to dequeue and output: O(|V|)
- Reduce In-Degree of all vertices adjacent to a vertex and Enqueue any In-Degree 0 vertices:
  - › O(|E|)
- For input graph G=(V,E) run time = O(|V| + |E|)
  - › Linear time!

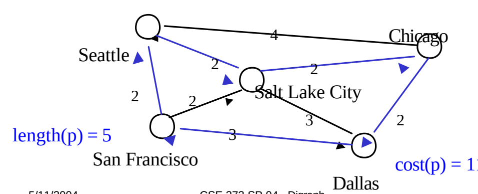
Topo Sort using a Stack (depth-first)

After each vertex is output, when updating In-Degree array, *push any vertex whose In-Degree becomes zero*



Recall Path cost ,Path length

- Path cost: the sum of the costs of each edge
- Path length: the number of edges in the path
  - › Path length is the unweighted path cost



Shortest Path Problems

- Given a graph G = (*V*, *E*) and a "source" vertex *s* in *V*, find the minimum cost paths from *s* to every vertex in *V*
- Many variations:
  - › unweighted vs. weighted
  - › cyclic vs. acyclic
  - › pos. weights only vs. pos. and neg. weights
  - › etc

Why study shortest path problems?

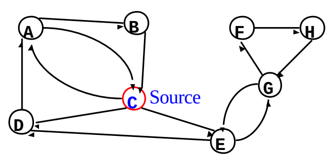
- Traveling on a budget: What is the cheapest airline schedule from Seattle to city X?
- Optimizing routing of packets on the internet:
  - › Vertices are routers and edges are network links with different delays. What is the routing path with smallest total delay?
- Shipping: Find which highways and roads to take to minimize total delay due to traffic
- etc.

Unweighted Shortest Path

Problem : Given a “source” vertex *s* in an unweighted directed graph

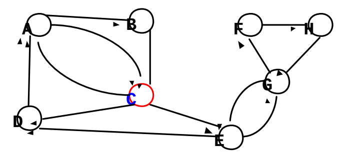
G = (*V*,*E*), find the shortest path from *s* to all vertices in G

Only interested in path lengths



Breadth-First Search Solution

- Basic Idea: Starting at node s, find vertices that can be reached using 0, 1, 2, 3, …, N-1 edges (works even for cyclic graphs!)



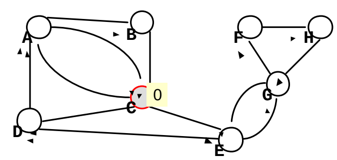
Breadth-First Search Alg.

- Uses a queue to track vertices that are “nearby”
- source vertex is **s**

```
Distance[s] := 0
Enqueue(Q,s); Mark(s)//After a vertex is marked once
                      // it won’t be enqueued again
while queue is not empty do
   X := Dequeue(Q);
   for each vertex Y adjacent to X do
      if Y is unmarked then
         Distance[Y] := Distance[X] + 1;
         Previous[Y] := X;//if we want to record paths
         Enqueue(Q,Y); Mark(Y);
```

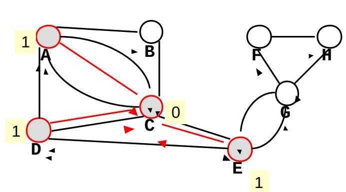
- Running time = O(|*V*| + |*E*|)

Example: Shortest Path length



Queue Q = C

Example (ct’d)

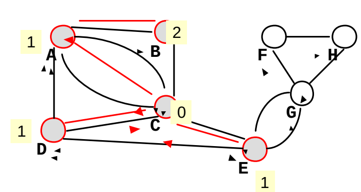


Previous pointer

Queue Q = A D E

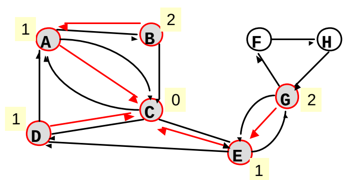
Indicates the vertex is marked

Example (ct’d)



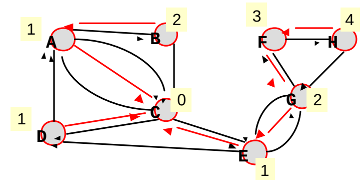
Q = D E B

Example (ct'd)



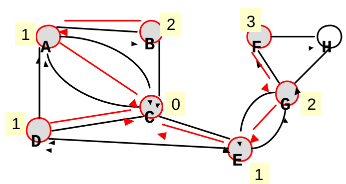
Q = B G

Example (ct'd)



Q = F

Example (ct'd)



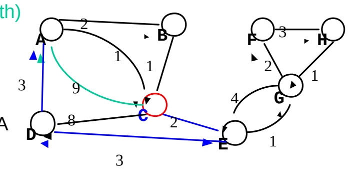
Q = H

What if edges have weights?

- Breadth First Search does not work anymore
  - › minimum *cost* path may have more edges than minimum *length* path

Shortest path (length) from C to A:
C→A (cost = 9)

Minimum Cost Path = C→E→D→A (cost = 8)



Dijkstra's Algorithm for Weighted Shortest Path

- Classic algorithm for solving shortest path in weighted graphs (without negative weights)
- A greedy algorithm (irrevocably makes decisions without considering future consequences)
- Each vertex has a cost for path from initial vertex

Basic Idea of Dijkstra's Algorithm

- Find the vertex with smallest cost that has not been "marked" yet.
- Mark it and compute the cost of its neighbors.
- Do this until all vertices are marked.
- Note that each step of the algorithm we are marking one vertex and we won't change our decision: hence the term "greedy" algorithm